

Solutions to Exercise #4

(範圍: Recurrence Relations)

1. P. 481: 1 (only for (a), (c) and (d)). (30%)

$$\text{Sol: (a) } a_n = a_{n-1} + 2n + 1 = (a_{n-2} + 2(n-1) + 1) + 2n + 1 = \dots = a_0 + \sum_{i=1}^n (2i+1)$$

$$= n^2 + 2n + 1 = (n+1)^2, \quad n \geq 0.$$

$$\text{(c) } a_n = 2a_{n-1} + 5 = 2(2a_{n-2} + 5) + 5 = \dots = 2^n a_0 + \sum_{i=1}^n (5 \cdot 2^{i-1}) = 6 \cdot 2^n - 5, \quad n \geq 0.$$

$$\text{(d) } a_n = 2a_{n-1} + 2^n.$$

$$a_n^h = c(2^n). \quad \text{Let } a_n^p = kn(2^n).$$

$$kn(2^n) - 2k(n-1)(2^{n-1}) = 2^{n-1}. \quad \Rightarrow \quad k = \frac{1}{2}.$$

$$a_n = a_n^h + a_n^p = c(2^n) + \frac{n}{2}(2^n).$$

$$a_0 = 1. \quad \Rightarrow \quad c = 1.$$

$$\text{Therefore, } a_n = 2^n + n \cdot 2^{n-1}, \quad n \geq 0.$$

2. P. 481: 6. (10%)

$$\text{Sol: } a_n - 6a_{n-1} + 9a_{n-2} = 3(2^{n-2}) + 7(3^{n-2}).$$

$$a_n^h = c_1(3^n) + c_2n(3^n). \quad \text{Let } a_n^p = k_1(2^n) + k_2n^2(3^n).$$

$$k_1(2^n) + k_2n^2(3^n) - 6[k_1(2^{n-1}) + k_2(n-1)^2(3^{n-1})] + 9[k_1(2^{n-2}) + k_2(n-2)^2(3^{n-2})]$$
$$= 3(2^{n-2}) + 7(3^{n-2})$$

$$\Rightarrow k_1 = 3, \quad k_2 = \frac{7}{18}.$$

$$a_n = a_n^h + a_n^p = (c_1 + c_2n + \frac{7}{18}n^2)(3^n) + 3(2^n).$$

$$a_0 = 1: \quad c_1 + 3 = 1.$$

$$a_1 = 4: \quad 3(c_1 + c_2 + \frac{7}{18}) + 6 = 4.$$

$$\Rightarrow c_1 = -2, \quad c_2 = \frac{17}{18}.$$

$$\text{Therefore, } a_n = (-2 + \frac{17}{18}n + \frac{7}{18}n^2)(3^n) + 3(2^n), \quad n \geq 0.$$

3. P. 481: 7. (10%)

$$\text{Sol: } a_n - 3a_{n-1} + 3a_{n-2} - a_{n-3} = 3 + 5(n-3).$$

$$a_n^h = c_1 + c_2n + c_3n^2. \quad \text{Let } a_n^p = k_1n^3 + k_2n^4.$$

$$k_1n^3 + k_2n^4 - 3[k_1(n-1)^3 + k_2(n-1)^4] + 3[k_1(n-2)^3 + k_2(n-2)^4] - [k_1(n-3)^3 + k_2(n-3)^4] = 3 + 5(n-3)$$

$$\Rightarrow k_1 = -\frac{3}{4}, \quad k_2 = \frac{5}{24}.$$

$$\text{Therefore, } a_n = a_n^h + a_n^p = c_1 + c_2n + c_3n^2 - \frac{3}{4}n^3 + \frac{5}{24}n^4, \quad n \geq 0.$$

4. P. 481: 8. (15%)

Sol: Let a_n denote the number of n -digit quaternary sequences in which no 3 appears to the right of a 0.

When the ending digit is 3, there are 3^{n-1} sequences that contain no 0.

When the ending digit is 0 (or 1 or 2), there are $3a_{n-1}$ sequences as required.

$$\Rightarrow a_n = 3a_{n-1} + 3^{n-1}, \quad a_0 = 1, \quad a_1 = 4.$$

$$a_n^h = c(3^n). \quad \text{Let } a_n^p = kn(3^n).$$

$$kn(3^n) = 3kn(3^{n-1}) + 3^{n-1} \Rightarrow k = \frac{1}{3}.$$

$$a_n = a_n^h + a_n^p = c(3^n) + n(3^{n-1}).$$

$$a_1 = 4: \quad 3c + 1 = 4 \Rightarrow c = 1.$$

$$\text{Therefore, } a_n = 3^n + n(3^{n-1}), \quad n \geq 0.$$

5. P. 487: 1 (only for (a) and (c)). (20%)

$$\text{Sol: (a) } a_n - a_{n-1} = 3^{n-1}. \quad \text{Let } A(x) = \sum_{n=0}^{\infty} a_n x^n.$$

$$(a_n - a_{n-1})x^n = 3^{n-1}x^n$$

$$\sum_{n=1}^{\infty} a_n x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = \sum_{n=1}^{\infty} 3^{n-1} x^n.$$

$$(A(x) - a_0) - xA(x) = \frac{x}{1-3x}.$$

$$A(x) = \frac{1}{1-x} + \frac{x}{(1-3x)(1-x)} = \frac{1}{2(1-x)} - \frac{1}{2(1-3x)}.$$

$$\text{Therefore, } a_n = \frac{1}{2} + \frac{1}{2}(3^n), \quad n \geq 0.$$

$$(c) \quad a_n - 3a_{n-1} + 2a_{n-2} = 0. \quad \text{Let } A(x) = \sum_{n=0}^{\infty} a_n x^n.$$

$$\sum_{n=2}^{\infty} a_n x^n - 3 \sum_{n=2}^{\infty} a_{n-1} x^n + 2 \sum_{n=2}^{\infty} a_{n-2} x^n = 0.$$

$$(A(x) - a_1 x - a_0) - 3x(A(x) - a_0) + 2x^2 A(x) = 0.$$

$$A(x) = \frac{1+3x}{(1-2x)(1-x)} = \frac{5}{1-2x} - \frac{4}{1-x}.$$

$$\text{Therefore, } a_n = 5(2^n) - 4, \quad n \geq 0.$$

6. Solve $a_n = 2na_{n-1} + n!$, $a_0 = 2$. (15%)

$$\text{Sol: } a_n^h = (n!)(2^n) a_0^h.$$

$$a_n = a_n^h \times b_n = (n!)(2^n) a_0^h \times b_n.$$

$$(n!)(2^n) a_0^h \times b_n = (2n) ((n-1)!)(2^{n-1}) a_0^h \times b_{n-1} + n!.$$

$$\Rightarrow b_n = b_{n-1} + \frac{1}{(2^n) a_0^h}$$

$$= b_0 + \left(\frac{1}{a_0^h}\right) \sum_{k=1}^n \frac{1}{2^k}$$

$$= \frac{2}{a_0^h} + \left(\frac{1}{a_0^h}\right) \left(1 - \frac{1}{2^n}\right) \quad (b_0 = \frac{2}{a_0^h})$$

$$= \left(\frac{1}{a_0^h}\right) \left(3 - \frac{1}{2^n}\right).$$

$$\begin{aligned}\Rightarrow a_n &= a_n^h \times b_n \\ &= (n!)(2^n) a_0^h \times \left(\frac{1}{a_0^h} \right) \left(3 - \frac{1}{2^n} \right) \\ &= (n!)(3 \times 2^n - 1).\end{aligned}$$