Solutions to Exercise #3

(範圍: Recurrence Relations)

1. P. 455: 2 (only for (c) and (d)). (12%)

Sol: (c) $a_n = \frac{4}{3} a_{n-1} = \left(\frac{4}{3}\right)^2 a_{n-2} = \left(\frac{4}{3}\right)^3 a_{n-3} = \ldots = \left(\frac{4}{3}\right)^{n-1} a_1 = \left(\frac{4}{3}\right)^{n-1} \times 5,\ n \geq 0.$

(d) $a_n = \frac{3}{2} a_{n-1} = \left(\frac{3}{2}\right)^2 a_{n-2} = \left(\frac{3}{2}\right)^3 a_{n-3} = \ldots = \left(\frac{3}{2}\right)^{n-4} a_4 = \left(\frac{3}{2}\right)^{n-4} \times 81$

$= \left(\frac{3}{2}\right)^n \times 16,\ n \geq 0.$

2. P. 468: 1 (only for (a), (c), (d) and (e)). (32%)

Sol: (a) Let $a_n = c \cdot r^n$.

characteristic equation: $r^2 - 5r - 6 = 0$.

characteristic roots: 6, -1.

general solution: $a_n = c_1 \cdot 6^n + c_2 \cdot (-1)^n$.

$a_0 = 1$: $c_1 + c_2 = 1$.
$a_1 = 3$: $6c_1 - c_2 = 3$.

$\Rightarrow c_1 = \frac{4}{7},\ \ c_2 = \frac{3}{7}$.

Therefore, $a_n = \frac{4}{7} \cdot 6^n + \frac{3}{7} \cdot (-1)^n,\ n \geq 0.$

(c) Let $a_n = c \cdot r^n$.

characteristic equation: $r^2 + 1 = 0$.

characteristic roots: $r_1 = i = \cos(\frac{\pi}{2}) + i\sin(\frac{\pi}{2}) = i\sin(\frac{\pi}{2})$ and

$r_2 = -i = \cos(-\frac{\pi}{2}) + i\sin(-\frac{\pi}{2}) = -i\sin(\frac{\pi}{2})$.

general solution: $a_n = c_1 \cdot \left(i\sin(\frac{n\pi}{2})\right) - c_2 \cdot \left(i\sin(\frac{n\pi}{2})\right) = k \cdot \sin(\frac{n\pi}{2})$, where $k = (c_1 - c_2)i$.

$a_1 = 3 \;\Rightarrow\; k = 3$.

Therefore, $a_n = 3\sin(\frac{n\pi}{2})$, $n \geq 0$.

(d) Let $a_n = c \cdot r^n$.

characteristic equation: $r^2 - 6r + 9 = 0$.

characteristic root: $r = 3$ (a root of multiplicity 2).

general solution: $a_n = c_1 \cdot 3^n + c_2 \cdot n \cdot 3^n$.

$a_0 = 5$: $c_1 = 5$.
$a_1 = 12$: $3c_1 + 3c_2 = 12$.

$\Rightarrow$ $c_1 = 5$, $c_2 = -1$.

Therefore, $a_n = 5 \cdot 3^n - n \cdot 3^n$, $n \geq 0$.

(e) Let $a_n = c \cdot r^n$.

haracteristic equation: $r^2 + 2r + 2 = 0$.

characteristic roots:

$r_1 = -1+i = \sqrt{2}\left(\cos(\frac{3\pi}{4}) + i\sin(\frac{3\pi}{4})\right)$ and

$r_2 = -1-i = \sqrt{2}\left(\cos(-\frac{3\pi}{4}) + i\sin(-\frac{3\pi}{4})\right) = \sqrt{2}\left(\cos(\frac{3\pi}{4}) - i\sin(\frac{3\pi}{4})\right)$.

general solution:

$$a_n = c_1 \cdot \left(\sqrt{2}\right)^n \left(\cos(\frac{3n\pi}{4}) + i\sin(\frac{3n\pi}{4})\right) + c_2 \cdot \left(\sqrt{2}\right)^n \left(\cos(\frac{3n\pi}{4}) - i\sin(\frac{3n\pi}{4})\right)$$

$$= \left(\sqrt{2}\right)^n \left(k_1 \cdot \cos(\frac{3n\pi}{4}) + k_2 \cdot \sin(\frac{3n\pi}{4})\right),$$

where $k_1 = c_1 + c_2$ and $k_2 = (c_1 - c_2)i$.

$a_0 = 1$: $k_1 = 1$.

$a_1 = 3$: $\sqrt{2}\left(-\frac{\sqrt{2}}{2}k_1 + \frac{\sqrt{2}}{2}k_2\right) = 3$.

$\Rightarrow$ $k_1 = 1$, $k_2 = 4$.

Therefore, $a_n = \left(\sqrt{2}\right)^n \left(\cos(\frac{3n\pi}{4}) + 4 \cdot \sin(\frac{3n\pi}{4})\right)$, $n \geq 0$.

3. Solve the following recurrence relations. (20%)

(a) $a_n + 5a_{n-1} + 8a_{n-2} + 4a_{n-3} = 0$, $a_1 = 0$, $a_2 = 1$, $a_3 = 3$, $n \geq 4$.

(b) $a_n - 5a_{n-1} + 7a_{n-2} - 3a_{n-3} = 0$, $a_0 = -1$, $a_1 = 1$, $a_2 = 3$, $n \geq 3$.

Sol: (a) Let $a_n = c \cdot r^n$.

characteristic equation: $r^3 + 5r^2 + 8r + 4 = 0$.

characteristic roots: -2 (a root of multiplicity 2), -1.

general solution: $a_n = c_1 \cdot (-2)^n + c_2 \cdot n \cdot (-2)^n + c_3 \cdot (-1)^n$.

$a_1 = 0$: $-2c_1 - 2c_2 - c_3 = 0$.

$a_2 = 1$: $4c_1 + 8c_2 + c_3 = 1$.

$a_3 = 3$: $-8c_1 - 24c_2 - c_3 = 3$.

$\Rightarrow$ $c_1 = 5$, $c_2 = -\dfrac{3}{2}$, $c_3 = -7$.

Therefore, $a_n = 5 \cdot (-2)^n - \dfrac{3n}{2} \cdot (-2)^n - 7 \cdot (-1)^n$, $n \geq 1$.

(b) Let $a_n = c \cdot r^n$.

characteristic equation: $r^3 - 5r^2 + 7r - 3 = 0$.

characteristic roots: 1 (a root of multiplicity 2), 3.

general solution: $a_n = c_1 + c_2 \cdot n + c_3 \cdot 3^n$.

$a_0 = -1$: $c_1 + c_3 = -1$.

$a_1 = 1$: $c_1 + c_2 + 3c_3 = 1$.

$a_2 = 3$: $c_1 + 2c_2 + 9c_3 = 3$.

$\Rightarrow$ $c_1 = -1$, $c_2 = 2$, $c_3 = 0$.

Therefore, $a_n = -1 + 2n$, $n \geq 0$.

4. P. 469: 9. (12%)

Sol: Let $a_n^{(1)}$ be the number of 1-2 sequences that sum to n and end with 1, and $a_n^{(2)}$ be the number of 1-2 sequences that sum to n and end with 2.

Then, $a_n = a_n^{(1)} + a_n^{(2)} = a_{n-1} + a_{n-2}$, where $n \geq 2$, $a_0 = 1$ and $a_1 = 1$.

$$\Rightarrow a_n = F_{n+1} = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right),\ n \geq 0.$$

5. P. 469: 12. (12%)

Sol: Let a_n: the number of ways to stack n poker chips that contain no consecutive blue chips;

$a_n^{(0)}$: the number of ways to stack n poker chips that contain no consecutive blue chips and end in blue;

$a_n^{(1)}$: the number of ways to stack n poker chips that contain no consecutive blue chips and end in red or white or green.

Then, $a_n = a_n^{(0)} + a_n^{(1)} = a_{n-1}^{(1)} + 3a_{n-1} = 3a_{n-2} + 3a_{n-1}$, where $n \geq 2$, $a_0 = 1$ and $a_1 = 4$.

Let $a_n = c \cdot r^n$..

characteristic equation: $r^2 - 3r - 3 = 0$.

characteristic roots: $\dfrac{3+\sqrt{21}}{2}$, $\dfrac{3-\sqrt{21}}{2}$.

general solution:

$$a_n = c_1 \cdot \left(\frac{3+\sqrt{21}}{2}\right)^n + c_2 \cdot \left(\frac{3-\sqrt{21}}{2}\right)^n.$$

$a_0 = 1$: $c_1 + c_2 = 1$.

$a_1 = 4$: $\dfrac{3+\sqrt{21}}{2} \cdot c_1 + \dfrac{3-\sqrt{21}}{2} \cdot c_2 = 4$.

$\Rightarrow$ $c_1 = \dfrac{5+\sqrt{21}}{2\sqrt{21}}$, $c_2 = \dfrac{\sqrt{21}-5}{2\sqrt{21}}$.

Therefore, $a_n = \dfrac{5+\sqrt{21}}{2\sqrt{21}} \cdot \left(\dfrac{3+\sqrt{21}}{2}\right)^n - \dfrac{5-\sqrt{21}}{2\sqrt{21}} \cdot \left(\dfrac{3-\sqrt{21}}{2}\right)^n$, $n \geq 0$.

6. P. 470: 24. (12%)

Sol: Clearly, $a_1 = 1$ and $a_2 = 3$.

When $n \geq 3$, let us consider the rightmost column of the chessboard.

If it is covered by one vertical (2×1) domino, then $a_n = a_{n-1}$.

If it is covered by two horizontal (1×2) dominos, then $a_n = a_{n-2}$.

If it is covered by one square (2×2) title, then $a_n = a_{n-2}$.

Hence, $a_n = a_{n-1} + 2\, a_{n-2}$.

characteristic equation: $r^2 - r - 2 = 0$.

characteristic roots: 2, -1.

general solution: $a_n = c_1 \cdot (-1)^n + c_2 \cdot 2^n$.

$a_1 = 1$: $(-1) \cdot c_1 + 2 \cdot c_2 = 1$.

$a_2 = 3$: $c_1 + 4 \cdot c_2 = 3$.

$\Rightarrow$ $c_1 = 1/3$, $c_2 = 2/3$.

Therefore, $a_n = (1/3) \cdot (-1)^n + (2/3) \cdot 2^n$, $n \geq 1$.