Solutions to Exercise #2

(範圍: Generating Functions)

1. P. 417: 2 (only for (c) and (e)). (10%)

Sol: (c) $(x^2 + x^3 + \ldots + x^{27})^5$.

(e) $(x^{10} + x^{11} + \ldots + x^{25})^2 \times (1 + x + x^2 + \ldots + x^{15})^3$.

2. P. 431: 12. (20%)

Sol: (a) The number of ways to distribute the 24 bottles of one type of soft drink among the surveyors so that each gets at least two bottles is equal to the coefficient of x^{24} in

$$\begin{aligned}(x^2 + x^3 + \ldots + x^{16})^5 &= x^{10} \times (1 + x + x^2 + \ldots + x^{14})^5 \\ &= x^{10} \times (1 + x + x^2 + \ldots)^5 \\ &= x^{10} \times (1 - x)^{-5},\end{aligned}$$

which is $\binom{-5}{14}(-1)^{14} = \binom{18}{14} = 3060$.

The number of ways to distribute the other type of soft drink is the same.

Therefore, the answer is $\binom{18}{14}^2 = 9363600$.

(b) If each surveyor gets at least three bottles of one type, then the number of ways for distribution is equal to the coefficient of x^{24} in

$$\begin{aligned}(x^3 + x^4 + \ldots + x^{12})^5 &= x^{15} \times (1 + x + x^2 + \ldots + x^9)^5 \\ &= x^{15} \times (1 + x + x^2 + \ldots)^5 \\ &= x^{10} \times (1 - x)^{-5},\end{aligned}$$

which is $\binom{-5}{9}(-1)^9 = \binom{13}{9} = 715$.

Thus, the answer is $\binom{18}{14}\binom{13}{9} = 2187900$.

3. P. 431: 16. (20%)

Sol: The number of ways to distribute the hamburgers is equal to the coefficient of x^{12} in

$$
\begin{aligned}
& (x + x^2 + \dots + x^6) \times (x^2 + x^3 + \dots + x^7)^3 \\
= \; & x^7 \times (1 + x + x^2 + \dots + x^5)^4 \\
= \; & x^7 \times (1 + x + x^2 + \dots)^4 \\
= \; & x^7 \times (1 - x)^{-4},
\end{aligned}
$$

which is $\binom{-4}{5}(-1)^5 = \binom{8}{5} = 56.$

The number of ways to distribute the hot dogs is equal to the coefficient of x^{16} in

$$
\begin{aligned}
& (x^3 + x^4 + \dots + x^{16}) \times (1 + x + x^2 + x^3 + x^4 + x^5)^3 \\
= \; & x^3 \times (1 + x + x^2 + \dots + x^{13}) \times (1 + x + x^2 + x^3 + x^4 + x^5)^3 \\
= \; & x^3 \times (1 + x + x^2 + \dots) \times (1 + x + x^2 + x^3 + x^4 + x^5)^3 \\
= \; & \frac{x^3}{1-x}\left(\frac{1-x^6}{1-x}\right)^3 \\
= \; & x^3 \times (1 - x^6)^3 \times (1 - x)^{-4} \\
= \; & x^3 \times \left[1 - \binom{3}{1}x^6 + \binom{3}{2}x^{12} - x^{18}\right] \times \left[1 + \binom{-4}{1}(-x) + \binom{-4}{2}(-x)^2 + \cdots\right],
\end{aligned}
$$

which is

$$
\begin{aligned}
& \binom{-4}{13}(-1)^{13} - \binom{3}{1}\binom{-4}{7}(-1)^7 + \binom{3}{2}\binom{-4}{1}(-1) \\
= \; & \binom{16}{13} - \binom{3}{1}\binom{10}{7} + \binom{3}{2}\binom{4}{1} \\
= \; & 212.
\end{aligned}
$$

Therefore, the answer is $\binom{8}{5} \times \left[\binom{16}{13} - \binom{3}{1}\binom{10}{7} + \binom{3}{2}\binom{4}{1}\right] = 11872.$

4. P. 440: 4 (only for (a) and (c)). (20%)

Sol: (a) $F(x) = (x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots)^4 = (e^x - 1)^4 = e^{4x} - 4e^{3x} + 6e^{2x} - 4e^x + 1.$

The answer is the coefficient of $\frac{x^{12}}{12!}$ in $F(x)$, which is

$4^{12} - 4 \times 3^{12} + 6 \times 2^{12} - 4 \times 1^{12} = 14676024.$

(c) When the numbers of blue flags and black flags are both even, the number of signals is the coefficient of $\frac{x^{12}}{12!}$ in the following generating function:

$$(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^{12}}{12!})^2 \times (1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + \frac{x^{12}}{12!})^2$$

$$= \quad (1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots)^2 \times (1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots)^2$$

$$= \quad e^{2x} \times \left(\frac{e^x + e^{-x}}{2}\right)^2$$

$$= \quad \frac{1}{4}e^{4x} + \frac{1}{2}e^{2x} + \frac{1}{4},$$

which is $\frac{4^{12}}{4} + \frac{2^{12}}{2}$.

When the numbers of blue flags and black flags are both odd, the number of signals is the coefficient of $\frac{x^{12}}{12!}$ in the following generating function:

$$(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^{10}}{10!})^2 \times (x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots + \frac{x^{11}}{11!})^2$$

$$= \quad (1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots)^2 \times (x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots)^2$$

$$= \quad e^{2x} \times \left(\frac{e^x - e^{-x}}{2}\right)^2$$

$$= \quad \frac{1}{4}e^{4x} - \frac{1}{2}e^{2x} + \frac{1}{4},$$

which is $\frac{4^{12}}{4} - \frac{2^{12}}{2}$.

Therefore, when the total number of blue flags and black flags is even, there are $\frac{4^{12}}{4}+\frac{2^{12}}{2}+\frac{4^{12}}{4}-\frac{2^{12}}{2}=\frac{4^{12}}{2}$ = 8388608 signals.

5. P. 440: 10 (only for (a), (b) and (d)). (30%)

Sol: (a) The answer is the coefficient of $\frac{x^{20}}{20!}$ in the following generating function:

$$(x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots+\frac{x^{19}}{19!})\times(x+\frac{x^3}{3!}+\frac{x^5}{5!}+\ldots+\frac{x^{19}}{19!})\times$$
$$(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots+\frac{x^{18}}{18!})^2$$

$$=(x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots)\times(x+\frac{x^3}{3!}+\frac{x^5}{5!}+\ldots)\times(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots)^2$$

$$=(e^x-1)\times\frac{e^x-e^{-x}}{2}\times e^{2x}$$

$$=\frac{1}{2}\times(e^{4x}-e^{3x}-e^{2x}+e^x),$$

which is $\frac{1}{2}\times(4^{20}-3^{20}-2^{20}+1)=548011897400.$

(b) The answer is the coefficient of $\frac{x^{20}}{20!}$ in the following generating function:

$$=(1+x+\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots+\frac{x^{20}}{20!})^4$$

$$=(1+x+\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots)^4$$

$$=(e^x-\frac{x^2}{2})^4$$

$$=e^{4x}-\binom{4}{1}e^{3x}\left(\frac{x^2}{2}\right)+\binom{4}{2}e^{2x}\left(\frac{x^2}{2}\right)^2-\binom{4}{3}e^{x}\left(\frac{x^2}{2}\right)^3+\left(\frac{x^2}{2}\right)^4,$$

which is $4^{20}-\binom{4}{1}\times\frac{3^{18}\times20\times19}{2}+\binom{4}{2}\times\frac{2^{16}\times20\times19\times18\times17}{4}$

$$- \binom{4}{3} \times \frac{1^{14} \times 20 \times 19 \times 18 \times 17 \times 16 \times 15}{8}$$

$$= 816488891656.$$

(d) The answer is the coefficient of $\frac{x^{20}}{20!}$ in the following generating function:

$$(1 + \frac{x^2}{2!}) \times (1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^{20}}{20!})^3$$

$$= (1 + \frac{x^2}{2!}) \times (1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots)^3$$

$$= (1 + \frac{x^2}{2!}) \times e^{3x},$$

which is $3^{20} + \frac{3^{18} \times 20 \times 19}{2} = 77096677311.$