Solutions to Exercise #1

(範圍: Principle of Inclusion and Exclusion)

1. Give the detailed computations of the right-hand side on page 14 of lecture notes. (10%)

Sol: Please refer to page 400 of Grimaldi's book.

2. P. 397: 6. (30%)

Sol: (a) $H\binom{4}{19}=\binom{4+19-1}{19}=\binom{22}{19}=1540$, where $H\binom{4}{19}$ is the number of ways to select 19 from 4 distinct objects, with repetitions allowed.

(b) For $1 \le i \le 4$, let c_i denote the condition of $x_i \ge 8$ (or equivalently, $x_i - 8 \ge 0$).

$N(c_i) = H\binom{4}{11} = \binom{14}{11} = 364$ is the number of nonnegative integer solutions to $x_1 + x_2 + x_3 + x_4 = 19 - 8$.

$N(c_i c_j) = H\binom{4}{3} = \binom{6}{3} = 20$ is the number of nonnegative integer solutions to $x_1 + x_2 + x_3 + x_4 = 19 - 8 \times 2$.

$N(c_i c_j c_k) = 0$.

$N(c_1 c_2 c_3 c_4) = 0$.

The answer to the problem is

$$N(\bar{c}_1\bar{c}_2\bar{c}_3\bar{c}_4) = N - \sum_{1\le i\le 4} N(c_i) + \sum_{1\le i<j\le 4} N(c_i c_j) = 204 .$$

(c) Let $y_1 = x_1$, $y_2 = x_2$, $y_3 = x_3 - 3$, and $y_4 = x_4 - 3$. The original problem is equivalent to $y_1 + y_2 + y_3 + y_4 = 13$ with $0 \le y_1 \le 5$, $0 \le y_2 \le 6$, $0 \le y_3 \le 4$ and $0 \le y_4 \le 5$.

Let c_1, c_2, c_3 and c_4 denote the conditions of $y_1 \ge 6$, $y_2 \ge 7$, $y_3 \ge 5$ and $y_4 \ge 6$, respectively.

$$N = H\binom{4}{13} = \binom{16}{13} = 560.$$

$$N(c_1) = N(c_4) = H\binom{4}{7} = \binom{10}{7} = 120.$$

$$N(c_2) = H\binom{4}{6} = \binom{9}{6} = 84.$$

$$N(c_3) = H\binom{4}{8} = \binom{11}{8} = 165.$$

$$N(c_1c_2) = 1.$$

$$N(c_1c_3) = H\binom{4}{2} = \binom{5}{2} = 10.$$

$$N(c_1c_4) = H\binom{4}{1} = \binom{4}{1} = 4.$$

$$N(c_2c_3) = H\binom{4}{1} = \binom{4}{1} = 4.$$

$$N(c_2c_4) = 1.$$

$$N(c_3c_4) = H\binom{4}{2} = \binom{5}{2} = 10.$$

$$N(c_ic_jc_k) = 0.$$

$$N(c_1c_2c_3c_4) = 0.$$

Then, $N(\bar{c}_1\bar{c}_2\bar{c}_3\bar{c}_4) = N - \sum_{1\le i\le 4} N(c_i) + \sum_{1\le i<j\le 4} N(c_ic_j) = 101.$

3. P. 397: 20. (10%)

Sol: Denote Sharon's seven friends by f_1, f_2, …, f_7. Let c_i be the condition that Sharon and f_i had lunch together. Then, the number of days Sharon had lunch alone is

$$N(\bar{c}_1\bar{c}_2\bar{c}_3\bar{c}_4\bar{c}_5\bar{c}_6\bar{c}_7) = 84 - \binom{7}{1}\cdot 35 + \binom{7}{2}\cdot 16 - \binom{7}{3}\cdot 8 + \binom{7}{4}\cdot 4 - \binom{7}{5}\cdot 2 + \binom{7}{6}\cdot 1 = 0.$$

4. P. 401: 2. (20%)

Sol: (a) Let c_1 (c_2, c_3, c_4, respectively) denote the condition of two consecutive A's (E's, N's R's) arranged.

$$N = (11!)/(2!)^4 = 2494800.$$

$$N(c_i) = (10!)/(2!)^3 = 453600.$$

$$N(c_i c_j) = (9!)/(2!)^2 = 90720.$$

$$N(c_i c_j c_k) = (8!)/(2!) = 20160.$$

$$N(c_1 c_2 c_3 c_4) = (7!) = 5040.$$

$$S_1 = \binom{4}{1} \cdot 453600 = 1814400.$$

$$S_2 = \binom{4}{2} \cdot 90720 = 544320.$$

$$S_3 = \binom{4}{3} \cdot 20160 = 80640.$$

$$S_4 = \binom{4}{4} \cdot 5040 = 5040.$$

$$E_2 = S_2 - \binom{3}{1} \cdot S_3 + \binom{4}{2} \cdot S_4 = 332640.$$

$$L_2 = S_2 - \binom{2}{1} \cdot S_3 + \binom{3}{1} \cdot S_4 = 398160.$$

(b) $$E_3 = S_3 - \binom{4}{1} \cdot S_4 = 60480.$$

$$L_3 = S_3 - \binom{3}{2} \cdot S_4 = 65520.$$

5. P. 403: 4. (10%)

Sol: The number of derangements of 1,2,3,4,5,6,7 is equal to

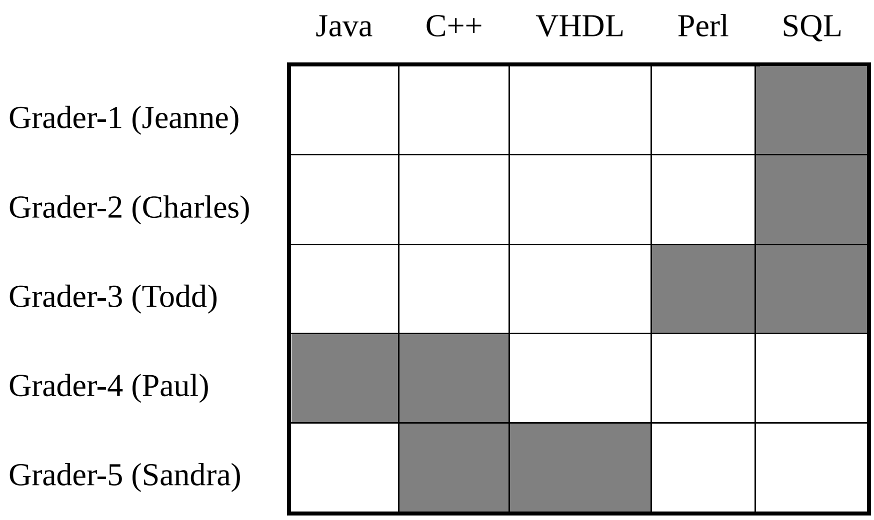
$$7! \times [1 - 1 + (1/2!) - (1/3!) + (1/4!) - (1/5!) + (1/6!) - (1/7!)] = 1854.$$

Therefore, there are $7! - 1854 = 3186$ permutations of 1, 2, 3, 4, 5, 6, 7 that are not derangements.

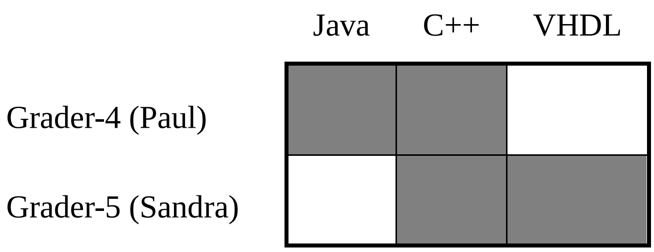
6. P. 410: 7. (20%)

Sol:

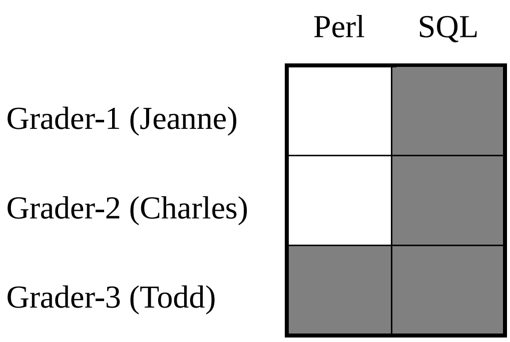
C:



C_1:



C_2:



$r(C_1, x) = 1 + 4x + 3x^2.$

$r(C_2, x) = 1 + 4x + 2x^2$.

$r(C, x) = (1 + 4x + 3x^2)(1 + 4x + 2x^2) = 1 + 8x + 21x^2 + 20x^3 + 6x^4$.

Let c_i be the condition of assigning Grader-i with a course that he or she dislikes.

$\boldsymbol{N(\bar{c}_1\bar{c}_2\bar{c}_3\bar{c}_4\bar{c}_5)} = 5! - 8 \times 4! + 21 \times 3! - 20 \times 2! + 6 \times 1! = 20.$