Solutions to Examination #1

(範圍: Combinatorics)

(For each problem (except problems 5, 10 and 12), please provide computation details, not the answer only.)

1. There are five books undergoing a two-round review process of five reviewers. Each book is reviewed by two distinct reviewers. In how many ways can these books be reviewed? (10%)

Sol: There are 5! ways for the first-round review and d_5 ways for the second-round review, where d_5 is the number of derangements of 1, 2, ..., 5. The latter is computed as follows.

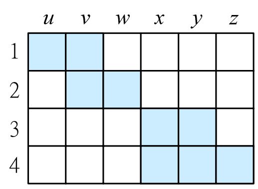
Let c_i denote the condition that integer i is at the ith place, where $1 \le i \le 5$. Then,

$$N(\overline{c}_1\overline{c}_2\overline{c}_3\overline{c}_4\overline{c}_5) = 5! - \binom{5}{1}4! + \binom{5}{2}3! - \binom{5}{3}2! + \binom{5}{4}1! - \binom{5}{5}0! = 44 .$$

Hence, the answer is $5! \times 44 = 5280$.

2. How many one-to-one functions f: {1, 2, 3, 4} $\to$ {u, v, w, x, y, z} are there so that $f(1) \notin \{u, v\}$, $f(2) \notin \{v, w\}$, $f(3) \notin \{x, y\}$, and $f(4) \notin \{x, y, z\}$? (10%)

Sol: Consider the following chessboard.



$$\begin{aligned} r(C, x) &= (1 + 4x + 3x^2) \times (1 + 5x + 4x^2) \\ &= 1 + 9x + 27x^2 + 31x^3 + 12x^4. \end{aligned}$$

Let c_i denote the condition that $(i, f(i))$ is located at a shaded square.

Then the answer is

$$N(\overline{c}_1\overline{c}_2\overline{c}_3\overline{c}_4) = (6\times5\times4\times3) - 9\times(5\times4\times3) + 27\times(4\times3) - 31\times3 + 12\times1 = 63 .$$

3. Determine how many integer solutions there are to $x_1 + x_2 + x_3 + x_4 = 19$, if $0 \le x_1 \le 5$, $0 \le x_2 \le 6$, $3 \le x_3 \le 7$ and $3 \le x_4 \le 8$. (10%)

Sol: The original problem is equivalent to $x_1 + x_2 + x_3 + x_4 = 13$ with $0 \le x_1 \le 5$, $0 \le x_2 \le 6$, $0 \le x_3 \le 4$ and $0 \le x_4 \le 5$.

Let c_1, c_2, c_3 and c_4 denote the conditions of $x_1 \ge 6$, $x_2 \ge 7$, $x_3 \ge 5$ and $x_4 \ge 6$, respectively.

$$N = H\binom{4}{13} = \binom{16}{13} = 560.$$

$$N(c_1) = N(c_4) = H\binom{4}{7} = \binom{10}{7} = 120. \quad N(c_2) = H\binom{4}{6} = \binom{9}{6} = 84.$$

$$N(c_3) = H\binom{4}{8} = \binom{11}{8} = 165.$$

$$N(c_1c_2) = 1. \quad N(c_1c_3) = H\binom{4}{2} = \binom{5}{2} = 10. \quad N(c_1c_4) = H\binom{4}{1} = \binom{4}{1} = 4.$$

$$N(c_2c_3) = H\binom{4}{1} = \binom{4}{1} = 4. \quad N(c_2c_4) = 1. \quad N(c_3c_4) = H\binom{4}{2} = \binom{5}{2} = 10.$$

$N(c_ic_jc_k) = 0$. $N(c_1c_2c_3c_4) = 0$.

Then, $N(\bar{c}_1\bar{c}_2\bar{c}_3\bar{c}_4) = N - \sum_{1\le i\le 4} N(c_i) + \sum_{1\le i<j\le 4} N(c_ic_j) = 101.$

4. Determine how many 4-element subsets of $S = \{1, 2, \ldots, 12\}$ contain no consecutive integers. (10%)

Sol: Let $\{a, b, c, d\}$ be an arbitrary 4-element subset, where $a < b < c < d$.

Define $c_1 = a-1$, $c_2 = b-a$, $c_3 = c-b$, $c_4 = d-c$, and $c_5 = 12-d$.

The problem is equivalent to finding the number of nonnegative integer solutions to $c_1 + c_2 + c_3 + c_4 + c_5 = 11$, where $c_1, c_5 \ge 0$, and $c_2, c_3, c_4 \ge 2$.

The answer is the coefficient of x^{11} in

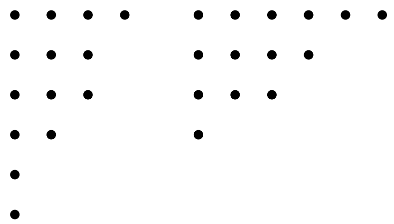
$$\begin{aligned} F(x) &= (1 + x + x^2 + \ldots + x^5)^2(x^2 + x^3 + \ldots + x^7)^3 \\ &= x^6(1 + x + x^2 + \ldots + x^5)^5 \\ &= x^6(1 + x + x^2 + \ldots + x^5 + \ldots)^5, \text{ or} \end{aligned}$$

the coefficient of x^5 in $(1 + x + x^2 + \ldots + x^5 + \ldots)^5$ $(= (1-x)^{-5})$,

which is equal to $\binom{-5}{5} \times (-1)^5 = 126$.

5. Explain why the number of partitions of 14 into 6 summands is identical with the number of partitions of 14 whose maximal summand is 6. (10%)

Sol: The main reason is that there is a one-to-one correspondence between a Ferrers graph and its transposition. Consider the following two Ferrers graph.



The left Ferrers graph represents $14 = 4 + 3 + 3 + 2 + 1 + 1$, a partition of 14 into 6 summands. The right Ferrers graph represents $14 = 6 + 4 + 3 + 1$, a partition of 14 with maximal summand 6. Both are the transposition of each other.

6. There are 4^{20} quaternary (0, 1, 2, 3) sequences of length 20, and $a \times 3^{20} + b \times 3^{18}$ of them each have exactly two 3's or none at all. Write (a, b). (10%)

Sol: The answer is the coefficient of $x^{20}/20!$ in the following generating function.

$$\begin{aligned} & (1 + x^2/2!) \times (1 + x + x^2/2! + x^3/3! + \ldots + x^{20}/20!)^3 \\ = \ & (1 + x^2/2!) \times (1 + x + x^2/2! + x^3/3! + \ldots)^3 \\ = \ & (1 + x^2/2!) \times e^{3x}, \end{aligned}$$

which is $3^{20} + (3^{18} \times 20 \times 19/2)$. Hence, $(a, b) = (1, 190)$.

7. Find the general solution for the recurrence relation $a_{n+3} - 3a_{n+2} + 3a_{n+1} - a_n = 3 + 5n$, where $n \geq 0$. (10%)

Sol: $a_n - 3a_{n-1} + 3a_{n-2} - a_{n-3} = 3 + 5(n - 3)$.

$a_n^h = c_1 + c_2 n + c_3 n^2$. Let $a_n^p = k_1 n^3 + k_2 n^4$.

$k_1 n^3 + k_2 n^4 - 3[k_1(n-1)^3 + k_2(n-1)^4] + 3[k_1(n-2)^3 + k_2(n-2)^4] - [k_1(n-3)^3 + k_2(n-3)^4] = 3 + 5(n-3)$

$\Rightarrow\ k_1 = -\dfrac{3}{4},\ \ k_2 = \dfrac{5}{24}$.

Hence, $a_n = a_n^h + a_n^p = c_1 + c_2 n + c_3 n^2 - \dfrac{3}{4} n^3 + \dfrac{5}{24} n^4$, where $n \geq 0$.

8. For $n \geq 0$, let a_n count the number of ways a sequence of 1's and 2's will sum to n. For example, $a_3 = 3$ because (1, 1, 1), (1, 2) and (2, 1) sum to 3. Find and solve a recurrence relation for a_n. (10%)

Sol: Let $a_n^{(1)}$ ($a_n^{(2)}$) be the number of 1-2 sequences that sum to n and end with 1 (2).

Then, $a_n = a_n^{(1)} + a_n^{(2)} = a_{n-1} + a_{n-2}$ for $n \geq 2$, $a_0 = 1$ and $a_1 = 1$.

$\Rightarrow$ $a_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right)$, where $n \geq 0$.

9. If you want to take out a loan of 10000 dollars with interest rate 2% per year and pay it back in 20 years, then you must make a constant payment $P = \alpha \times (\beta - \gamma^{-20})^{-1}$ at the end of each year. Write (α, β, γ). (10%)

Sol: Let a_n be the amount you still owed at the end of the nth year.

When $1 \leq n \leq 20$, we have $a_n = a_{n-1} + 0.02 \times a_{n-1} - P = (1.02) \times a_{n-1} - P$

with $a_0 = 10000$ and $a_{20} = 0$.

$\Rightarrow$ $a_n^h = c \times (1.02)^n$ and $a_n^p = d$.

$\Rightarrow$ $d = (1.02) \times d - P$

$\Rightarrow$ $d = P/0.02$

$\Rightarrow$ $a_n = a_n^h + a_n^p = c \times (1.02)^n + P/0.02$

Since $a_0 = 10000$, we have $c + P/0.02 = 10000$, i.e., $c = 10000 - P/0.02$.

$\Rightarrow$ $a_n = (10000 - P/0.02) \times (1.02)^n + P/0.02$

Again, since $a_{20} = 0$, we have $P = 200 \times (1 - (1.02)^{-20})^{-1}$.

Hence, $(\alpha, \beta, \gamma) = (200, 1, 1.02)$.

10. Let a_n be the number of distinct outputs that may be generated from a stack with n consecutive integers as inputs. Find a recurrence relation for a_n and explain your answer. (10%)

Sol: Suppose that $1, 2, \ldots, n$ are the inputs of the stack. So, each output of the stack is a permutation of $1, 2, \ldots, n$.

Consider an arbitrary output $x_1, x_2, \ldots, x_{i-1}, 1, x_{i+1}, x_{i+2}, \ldots, x_n$ of the stack, where $1 \leq i \leq n$.

We have $\{x_1, x_2, \ldots, x_{i-1}\} = \{2, 3, \ldots, i\}$ ($= \varnothing$ as $i = 1$) and

$\{x_{i+1}, x_{i+2}, \ldots, x_n\} = \{i+1, i+2, \ldots, n\}$ ($= \varnothing$ as $i = n$).

$\Rightarrow$ $a_n = a_0 a_{n-1} + a_1 a_{n-2} + \ldots + a_{n-2} a_1 + a_{n-1} a_0$ for $n \geq 1$ and $a_0 = 1$.

11. Let $f(x)$ be the ordinary generating function of $(a_0, a_1, a_2, \ldots, a_n, \ldots)$, where $a_0 = 1$ and $a_n = 2a_{n-1} + 3$ for $n \geq 1$. Find b_n so that $(1 - x)f(x)$ is the ordinary generating function of $(b_0, b_1, b_2, \ldots, b_n, \ldots)$. (hint: if $h(x)$ is the ordinary generating function of

$(c_0, c_1, c_2, \ldots, c_n, \ldots)$, then $h(x)/(1-x)$ is the ordinary generating function of $(c_0, c_0 + c_1, c_0 + c_1 + c_2, \ldots))$. (10%)

Sol: Solving the recurrence relation $a_n = 2a_{n-1} + 3$, we have $a_n = 2^{n+2} - 3$, where $n \geq 0$.
Let $g(x) = (1-x)f(x)$, and we have $f(x) = g(x)/(1-x)$, which is the ordinary generating function of $(b_0, b_0 + b_1, b_0 + b_1 + b_2, \ldots)$.
Hence, we have $b_0 = a_0 = 1$ and for $n \geq 1$,

$$\begin{aligned} a_n &= b_0 + b_1 + \ldots + b_n \\ &= a_{n-1} + b_n \end{aligned}$$

i.e.,

$$\begin{aligned} b_n &= a_n - a_{n-1} \\ &= (2^{n+2} - 3) - (2^{n+1} - 3) \\ &= 2^{n+1}. \end{aligned}$$

12. An arrangement of 1, 2, …, n is called a *derangement*, if each number is not at its natural position, i.e., 1 is not at the first place, 2 is not at the second place, ..., and n is not at the nth place. Let d_n be the number of derangements of 1, 2, ..., n. Clearly, we have $d_1 = 0$ and $d_2 = 1$. Show that when $n > 2$, we have $d_n = (n-1)(d_{n-1} + d_{n-2})$. (hint: first consider the location of 1, say the ith place ($i \neq 1$), and then consider the location of i.) (10%)

Sol. When 1 is at the ith place ($i \neq 1$), the value of d_n can be calculated according to the following two cases.

Case 1. i is at the first place.

There are d_{n-2} derangements.

Case 2. i is not at the first place.

Considering the first place the natural position of i, we have $d_n = d_{n-1}$.

Since there are $n-1$ possible locations for 1, we have

$$d_n = (n-1)(d_{n-1} + d_{n-2}) \text{ for } n > 2.$$