Principle of Inclusion and Exclusion

• An Illustrative Example

Determine the number of integers n, $1 \le n \le 100$, which are not divisible by 2, 3, 5.

Let $S = \{1, 2, \ldots, 100\}$ and $N = |S| = 100$.

Define three conditions as follows:

- c_1 : **divisible by 2;**
- c_2 : **divisible by 3;**
- c_3 : **divisible by 5.**

The answer is denoted by $N(\bar{c}_1\, \bar{c}_2\, \bar{c}_3)$, which is evaluated below.

$N(c_1) = \lfloor 100/2 \rfloor = 50$; $N(c_2) = \lfloor 100/3 \rfloor = 33$;

$N(c_3) = \lfloor 100/5 \rfloor = 20$.

$N(c_1c_2) = \lfloor 100/6 \rfloor = 16$; $N(c_1c_3) = \lfloor 100/10 \rfloor = 10$;

$N(c_2c_3) = \lfloor 100/15 \rfloor = 6$.

$N(c_1c_2c_3) = \lfloor 100/30 \rfloor = 3$.

$$
\begin{aligned}
N(\bar{c}_1\,\bar{c}_2\,\bar{c}_3) &= N - (N(c_1) + N(c_2) + N(c_3)) + \\
&\quad (N(c_1c_2) + N(c_1c_3) + N(c_2c_3)) - \\
&\quad N(c_1c_2c_3) \\
&= 100 - (50 + 33 + 20) + (16 + 10 + 6) - 3 \\
&= 26.
\end{aligned}
$$

• The Principle

S : a set; $N = |S|$

$c_1, c_2, \ldots, c_t$: conditions

$N(c_i)$: the number of elements in S that satisfy c_i

$N(c_i c_j)$: the number of elements in S that satisfy c_i and c_j (and perhaps some others)

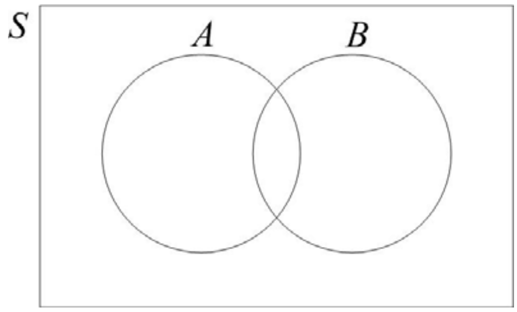
$N(\overline{c}_i)$: the number of elements in S that do not satisfy c_i $(N(\overline{c}_i) = N - N(c_i))$

$N(\overline{c}_i\, \overline{c}_j)$: the number of elements in S that do not satisfy either of c_i and c_j

Theorem.* *The number of elements in S that satisfy none of $c_1, c_2, \ldots, c_t$ is equal to

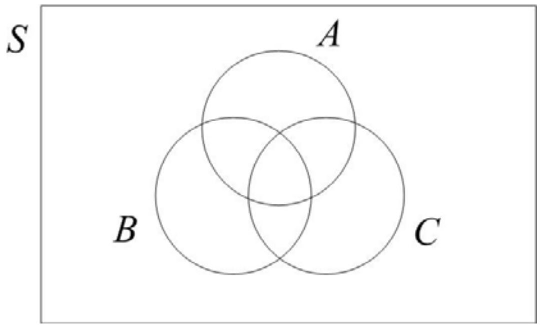
$$N(\bar{c}_1\,\bar{c}_2\ldots\bar{c}_t) \quad = \quad N \; - \sum_{1\le i\le t} N(c_i) \; + \sum_{1\le i<j\le t} N(c_ic_j) \; - \sum_{1\le i<j<k\le t} N(c_ic_jc_k) \; + \; \cdots \; + \; (-1)^t N(c_1c_2\ldots c_t).$$

When $t=2$, $c_1 \to A$ and $c_2 \to B$.



$$
\begin{aligned}
N(\bar{c}_1\,\bar{c}_2) &= |S| - |A \cup B| \\
&= |S| - (|A| + |B| - |A \cap B|) \\
&= |S| - (|A| + |B|) + |A \cap B|
\end{aligned}
$$

When $t = 3$, $c_1 \rightarrow A$, $c_2 \rightarrow B$, and $c_3 \rightarrow C$.

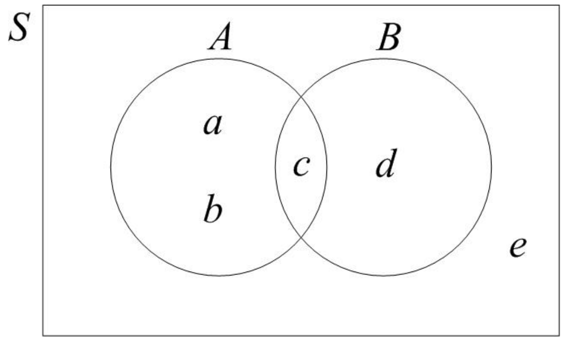


$$\begin{aligned} N(\bar{c}_1\,\bar{c}_2\,\bar{c}_3) &= |S| - |A \cup B \cup C| \\ &= |S| - (|A| + |B| + |C| - |A \cap B| - |A \cap C| - \\ &\quad |B \cap C| + |A \cap B \cap C|) \\ &= |S| - (|A| + |B| + |C|) + \\ &\quad (|A \cap B| + |A \cap C| + |B \cap C|) - \\ &\quad |A \cap B \cap C| \end{aligned}$$

This theorem can be proved by induction on *t*.

Here we prove it by a combinatorial method.

Observation :



$$|A \cup B| = |A| + |B| - |A \cap B|$$

a	**1**	**=**	**1**	**+**	**0**	**–**	**0**
b	**1**	**=**	**1**	**+**	**0**	**–**	**0**
c	**1**	**=**	**1**	**+**	**1**	**–**	**1**
d	**1**	**=**	**0**	**+**	**1**	**–**	**0**
e	**0**	**=**	**0**	**+**	**0**	**–**	**0**
	4	**=**	**3**	**+**	**2**	**–**	**1**

Proof : For each $x \in S$, we show that x contributes the same count to each side of the equation.

$$N(\bar{c}_1 \bar{c}_2 \ldots \bar{c}_t) = N - \sum_{1 \le i \le t} N(c_i) + \sum_{1 \le i < j \le t} N(c_i c_j) - \sum_{1 \le i < j < k \le t} N(c_i c_j c_k) + \ldots + (-1)^t N(c_1 c_2 \ldots c_t).$$

Case 1. x **satisfies none of the conditions.**

x **is counted once in** $N(\bar{c}_1 \bar{c}_2 \ldots \bar{c}_t)$ **and** N, **but not in the other terms.**

Case 2. x **satisfies** r **of the conditions.**

(1) x contributes nothing to $N(\bar{c}_1\,\bar{c}_2 \ldots \bar{c}_t)$

(2) x is counted once in N.

(3) x is counted $\binom{r}{1}$ times in $\sum_{1 \le i \le t} N(c_i)$.

(4) x is counted $\binom{r}{2}$ times in $\sum_{1 \le i < j \le t} N(c_i c_j)$.

$\vdots$

x is counted $\binom{r}{r}$ times in $\sum N(c_{i_1}\, c_{i_2} \ldots\, c_{i_r})$.

$\Rightarrow$ left-hand side : 0.

right-hand side : $1 - \binom{r}{1} + \binom{r}{2} - \ldots + (-1)^r \binom{r}{r}$

$= (1 + (-1))^r = 0.$

Corollary. ***The number of elements in S that satisfy at least one of the conditions is*** $N(c_1 \text{ or } c_2 \text{ or } \ldots \text{ or } c_t) = N - N(\bar{c}_1\,\bar{c}_2 \ldots \bar{c}_t)$.

Let $S_0 = N$.

$$S_1 = \sum_{1 \le i \le t} N(c_i).$$

$$S_2 = \sum_{1 \le i < j \le t} N(c_i c_j).$$

$$\vdots$$

$$S_k = \sum N(c_{i_1} c_{i_2} \dots c_{i_k}).$$

Theorem. ***The number of elements in S that satisfy exactly m of t conditions is***

$$E_m = S_m - \binom{m+1}{1} S_{m+1} + \binom{m+2}{2} S_{m+2} - \dots + (-1)^{t-m} \binom{t}{t-m} S_t.$$

Proof. **For each $x \in S$, we show that x contributes the same count to each side of the equation.**

Case 1. **x satisfies fewer than m conditions.**

x is not counted on either side of the equation.

Case 2. **x satisfies exactly m of the conditions.**

x is counted once in E_m and once in S_m, but not in S_{m+1}, S_{m+2}, …, S_t.

Case 3. x **satisfies** r ($m < r \le t$) **of the conditions.**

x **is counted** $\binom{r}{m}$ **times in** S_m, $\binom{r}{m+1}$ **times in** S_{m+1}, …, $\binom{r}{r}$ **times in** S_r.

But, x **contributes nothing to the other terms.**

$\Rightarrow$ **left-hand side : 0.**

right-hand side :

$$\binom{r}{m} - \binom{m+1}{1}\binom{r}{m+1} + \binom{m+2}{2}\binom{r}{m+2} - \dots + (-1)^{r-m}\binom{r}{r-m}\binom{r}{r}$$

$=$

⋮ **(left as an exercise)**

$= 0.$

Theorem. The number of elements in S that satisfy at least m of t conditions is

$$L_m = S_m - \binom{m}{m-1} S_{m+1} + \binom{m+1}{m-1} S_{m+2} - \ldots + (-1)^{t-m} \binom{t-1}{m-1} S_t.$$

An inductive proof of this theorem was outlined as problem 8 on page 401 of Grimaldi's book.

For your reference, the following shows an alternative proof that is based on a combinatorial method.

For each $x \in S$, we show that x contributes the same count to each side of the equation.

Case 1. x satisfies fewer than m conditions.

x is not counted on either side of the equation.

Case 2. x satisfies r ($m \le r \le t$) of the t conditions.

x is counted once in L_m, $\binom{r}{m}$ times in S_m, $\binom{r}{m+1}$ times in S_{m+1}, …, $\binom{r}{r}$ times in S_r. But, x contributes nothing to any of the other terms in the equation.

Left-hand side: 1.

Right-hand side:

$$\binom{r}{m} - \binom{m}{m-1}\binom{r}{m+1} + \binom{m+1}{m-1}\binom{r}{m+2} - \ldots + (-1)^{r-m}\binom{r-1}{m-1}\binom{r}{r}$$

$$= \sum_{0 \le k \le r-m} \binom{r}{m+k}\binom{(m-1)+k}{m-1}(-1)^k$$

$$= \sum_{0 \le k \le r-m} \binom{r}{m+k}\binom{(m-1)+k}{k}(-1)^k \quad \text{(assume } m-1+k \ge 0\text{)}$$

$$= \sum_{0 \le k \le r-m} \binom{r}{m+k}\binom{-m}{k}$$

(refer to *Concrete Mathematics*, 2nd edition, by Graham, Knuth, and Patashnik, pp. 164, Eq. (5.14))

$$= \sum_{k} \binom{r}{m+k}\binom{-m}{k} \quad \left(\binom{-m}{k} = 0 \text{ as } k < 0, \text{ and } \binom{r}{m+k} = 0 \text{ as } k > r-m\right)$$

$$= \binom{r-m}{r-m}$$

(refer to *Concrete Mathematics*, 2nd edition, by Graham, Knuth, and Patashnik, pp. 169, Eq. (5.23))

$$= 1.$$

Since $m \ge 0$ and $k \ge 0$, we have $m = 0$ and $k = 0$ if $m - 1 + k < 0$.

When $m = 0$ and $k = 0$, the right-hand side is $\binom{r}{0} = 1$.

Ex. Compute the number of integer solutions to

$x_1 + x_2 + x_3 + x_4 = 18$, where $0 \le x_i \le 7$ for $1 \le i \le 4$.

Let S be the set of integer solutions to

$x_1 + x_2 + x_3 + x_4 = 18$, where $x_i \ge 0$ for $1 \le i \le 4$.

Also, let c_i denote the constraint of $x_i \ge 8$.

Then, $N(\overline{c_1}\,\overline{c_2}\,\overline{c_3}\,\overline{c_4})$ is the answer.

$N = H(4, 18) = C(4+18-1, 18) = C(21, 18)$.

$N(c_i) = H(4, 10) = C(13, 10)$.

$N(c_ic_j) = H(4, 2) = C(5, 2)$.

$N(c_ic_jc_k) = 0$.

$N(c_1c_2c_3c_4) = 0$.

$$N(\overline{c_1}\,\overline{c_2}\,\overline{c_3}\,\overline{c_4}) = C(21, 18) - C(4, 1) \times C(13, 10) + C(4, 2) \times C(5, 2) - 0 + 0 = 246.$$

Ex. Compute the number of ways to permute *a*, *b*, *c*, …, *y*, *z* so that none of *car*, *dog*, *pun*, and *byte* occurs.

Let *S* be the set of all permutations of the 26 letters.

Let c_1, c_2, c_3 and c_4 denote the conditions that the permutations contain *car*, *dog*, *pun*, and *byte*, respectively.

Then, $N(\overline{c_1}\,\overline{c_2}\,\overline{c_3}\,\overline{c_4})$ is the answer.

$N = 26!$.

$N(c_1) = N(c_2) = N(c_3) = 24!$; $N(c_4) = 23!$.

$N(c_1c_2) = N(c_1c_3) = N(c_2c_3) = 22!$; $N(c_ic_4) = 21!$.

$N(c_1c_2c_3) = 20!$; $N(c_ic_jc_4) = 19!$.

$N(c_1c_2c_3c_4) = 17!$.

$$N(\overline{c_1}\,\overline{c_2}\,\overline{c_3}\,\overline{c_4}) = 26! - (3\times 24! + 23!) + (3\times 22! + 3\times 21!) - (20! + 3\times 19!) + 17!.$$

Ex. **Given a positive integer $n \geq 2$, the *Euler's phi function*, denoted by $\phi(n)$, is the number of integers m so that $1 \leq m \leq n$ and $\gcd(m, n) = 1$ (m, n are relatively prime). For example, $\phi(2) = 1$, $\phi(3) = 2$, $\phi(4) = 2$, and $\phi(5) = 4$.**

Consider $n = 1260 = 2^2 \times 3^2 \times 5 \times 7$ (質因數分解).

Let $S = \{1, 2, \ldots, 1260\}$.

$\phi(1260)$ can be computed as follows.

Let c_1, c_2, c_3, c_4 denote the conditions that the integers m are divisible by 2, 3, 5, 7, respectively.

Then, $\phi(1260) = N(\overline{c_1}\,\overline{c_2}\,\overline{c_3}\,\overline{c_4})$.

$N = 1260.$

$N(c_1) = 1260/2 = 630;\quad N(c_2) = 1260/3 = 420;$

$N(c_3) = 1260/5 = 252;\quad N(c_4) = 1260/7 = 180.$

$N(c_1c_2) = 1260/6 = 210;\quad N(c_1c_3) = 1260/10 = 126;$

$N(c_1c_4) = 1260/14 = 90;\quad N(c_2c_3) = 1260/15 = 84;$

$N(c_2c_4) = 1260/21 = 60;\quad N(c_3c_4) = 1260/35 = 36.$

$N(c_1c_2c_3) = 1260/30 = 42;\quad N(c_1c_2c_4) = 1260/42 = 30;$

$N(c_1c_3c_4) = 1260/70 = 18;\quad N(c_2c_3c_4) = 1260/105 = 12.$

$N(c_1c_2c_3c_4) = 1260/210 = 6.$

$$\begin{aligned} N(\overline{c}_1\,\overline{c}_2\,\overline{c}_3\,\overline{c}_4) &= 1260 - (630+420+252+180) + \\ &\quad (210+126+90+84+60+36) - \\ &\quad (42+30+18+12) + 6 \\ &= 288. \end{aligned}$$

In general, suppose $n = p_1^{e_1}\, p_2^{e_2} \dots p_t^{e_t}$ (質因數分解).

Then, $\phi(n) = n \times (1 - \frac{1}{p_1}) \times (1 - \frac{1}{p_2}) \times \dots \times (1 - \frac{1}{p_t})$.

For example, when $n = 1260$,

$$\phi(1260) = 1260 \times (1 - \frac{1}{2}) \times (1 - \frac{1}{3}) \times (1 - \frac{1}{5}) \times (1 - \frac{1}{7})$$

$$= 288.$$

Another description of Euler's formula (? no proof was found):

Suppose that n is a positive integer and p_1, p_2, …, p_r are distinct prime numbers, where $n > p_i$ for all $1 \leq i \leq r$.

Let $F(n) = |\{m \mid 1 \leq m \leq n$ is an integer and $\gcd(m, p_i) = 1$ for all $1 \leq i \leq r\}|$.

Then, $F(n)$ is equal to

$$f(n) = n \times (1 - \frac{1}{p_1}) \times (1 - \frac{1}{p_2}) \times \ldots \times (1 - \frac{1}{p_r}),$$

if $f(n)$ is an integer.

Ex. **Suppose that $n = 23$, $p_1 = 2$, and $p_2 = 3$ (i.e., $r = 2$).**

There are 8 integers in {1, 2, …, 23} that are relatively prime to both 2 and 3. They are 1, 5, 7, 11, 13, 17, 19, and 23.

$$f(n) = n \times (1 - \frac{1}{2}) \times (1 - \frac{1}{3}) = \frac{n}{3}.$$

$f(23)$ is not an integer

We first compute $f(21) = 7$ (or $f(24) = 8$), and then obtain $F(23) = F(21) + 1 = f(21) + 1 = 8$ (or $F(23) = F(24) = f(24) = 8$), where 1 indicates the integer 23.

Ex. Six married couples are to be seated at a circular table. In how many ways can they be arranged so that no wife sits next to her husband?

Let c_i denote the condition that couple i are neighboring, where $1 \le i \le 6$.

The answer is $N(\overline{c_1}\,\overline{c_2}\ldots\overline{c_6})$.

$N = \frac{12!}{12} = 11!.$

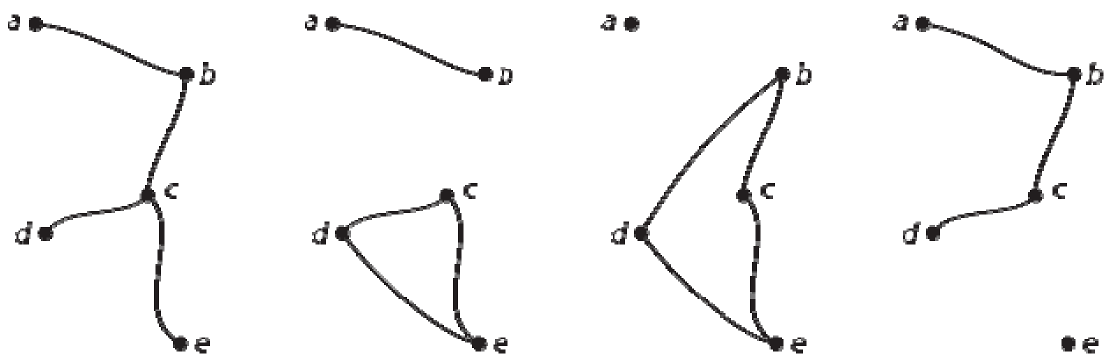
$N(c_i) = \frac{11!}{11} \times 2 = 2 \times 10!.$

$N(c_ic_j) = \frac{10!}{10} \times 2^2 = 2^2 \times 9!.$

Similarly, $N(c_ic_jc_k) = 2^3 \times 8!$, $N(c_ic_jc_kc_l) = 2^4 \times 7!$, $N(c_ic_jc_kc_lc_r) = 2^5 \times 6!$, $N(c_ic_jc_kc_lc_rc_s) = 2^6 \times 5!$.

$$\begin{aligned} N(\overline{c_1}\,\overline{c_2}\ldots\overline{c_6}) &= 11! - C(6, 1) \times 2 \times 10! + C(6, 2) \times 2^2 \times 9! \\ &\quad - C(6, 3) \times 2^3 \times 8! + C(6, 4) \times 2^4 \times 7! \\ &\quad - C(6, 5) \times 2^5 \times 6! + C(6, 6) \times 2^6 \times 5! \\ &= 12{,}771{,}840. \end{aligned}$$

Ex. How many ways are there to connect five vertices *a*, *b*, *c*, *d* and *e* so that none of them is isolated?



(O) (O) (X) (X)

Let c_1, c_2, c_3, c_4 and c_5 denote the conditions that vertices *a*, *b*, *c*, *d* and *e* are isolated, respectively.

The answer is $N(\overline{c}_1\,\overline{c}_2\ldots\overline{c}_5)$.

$N = 2^{10}$ **(at most 10 edges among five vertices).**

$N(c_i) = 2^6$.

$N(c_i c_j) = 2^3$.

$N(c_i c_j c_k) = 2^1$.

$N(c_i c_j c_k c_l) = N(c_i c_j c_k c_l c_r) = 2^0$.

$$
\begin{aligned}
N(\overline{c}_1\,\overline{c}_2 \ldots \overline{c}_5) &= 2^{10} - C(5,1) \times 2^6 + C(5,2) \times 2^3 \\
&\quad - C(5,3) \times 2^1 + C(5,4) \times 2^0 \\
&\quad - C(5,5) \times 2^0 \\
&= 768.
\end{aligned}
$$

Ex. For the example above, how many ways are there to connect five vertices a, b, c, d and e so that exactly two of them are isolated?

The answer is

$$
\begin{aligned}
E_2 &= S_2 - C(3, 1)\times S_3 + C(4, 2)\times S_4 \\
&\quad - C(5, 3)\times S_5 \\
&= C(5, 2)\times 2^3 - C(3, 1)\times C(5, 3)\times 2^1 \\
&\quad + C(4, 2)\times C(5, 4)\times 2^0 \\
&\quad - C(5, 3)\times C(5, 5)\times 2^0 \\
&= 80 - 60 + 30 - 10 \\
&= 40.
\end{aligned}
$$

• Derangements

An arrangement of 1, 2, …, n is called a *derangement*, if 1 is not at the first place (its natural position), 2 is not at the second place (its natural position), …, and n is not at the nth place (its natural position).

Consider $n = 10$, and let c_i denote the condition that integer i is at the ith place, where $1 \leq i \leq 10$.

The number of derangements of 1, 2, …, 10 is

$$N(\bar{c}_1 \bar{c}_2 \ldots \bar{c}_{10}) = 10! - \binom{10}{1}9! + \binom{10}{2}8! - \ldots + \binom{10}{10}0!$$

$$= 1334960.$$

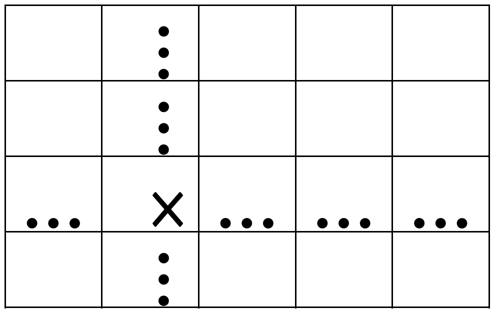
Ex. There are seven books undergoing a two-round review process of seven reviewers. Each book is reviewed by two distinct reviewers. In how many ways can these books be reviewed?

There are 7! ways for the first-round review.

There are d_7 ways for the second-round review, where d_7 is the number of derangements of 1, 2, …, 7.

The answer is $7! \times d_7$.

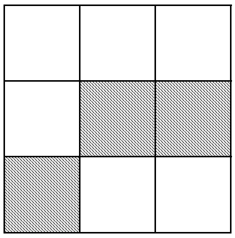
• Rook Polynomials



× : **rook (車)**

Problem : Given a chessboard *C* of arbitrary shape and size, determine the number r_k of ways of placing *k* nontaking rooks on *C*.

C :



$r_1 = 6, \quad r_2 = 8, \quad r_3 = 2$

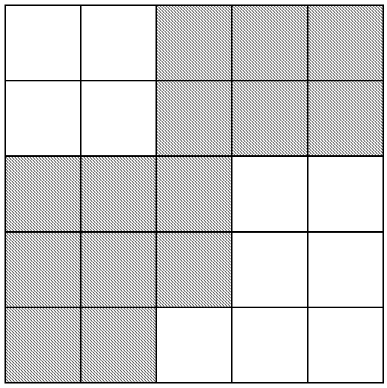
$r_k = 0 \quad \text{for } k \geq 4$

Let $r_0 = 1$.

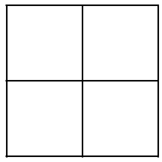
rook polynomial $R(C, x) = \sum_{i=0}^{\infty} r_i x^i$

$= 1 + 6x + 8x^2 + 2x^3$

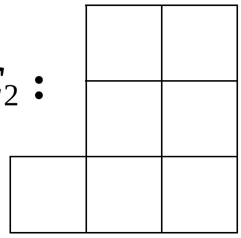
C :



C_1 :



C_2 :



$$R(C, x) = 1 + 11x + 40x^2 + 56x^3 + 28x^4 + 4x^5$$

$$R(C_1, x) = 1 + 4x + 2x^2$$

$$R(C_2, x) = 1 + 7x + 10x^2 + 2x^3$$

C_1 and C_2 are "disjoint" (no square in the same row or column)

$$\Rightarrow \quad R(C, x) = R(C_1, x) \cdot R(C_2, x)$$

Count r_3 in C as follows.

Case 1. All three rooks are placed on C_2: 2 ways

Case 2. Two rooks are placed on C_2 and one rook is placed on C_1: $10 \times 4 = 40$ ways

Case 3. One rook is placed on C_2 and two rooks are placed on C_1: $7 \times 2 = 14$ ways

$$r_3 = 2 + 40 + 14 = 56$$

Compute the coefficient r_3 of x^3 in $R(C_1, x) \cdot R(C_2, x)$:

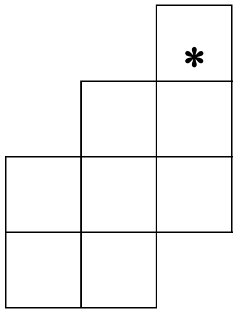
$$
\begin{aligned}
& R(C_1, x) \cdot R(C_2, x) \\
= \ & (1 + 4x + 2x^2) \cdot (1 + 7x + 10x^2 + 2x^3) \\
= \ & \ldots + 1 \cdot 2x^3 + 4x \cdot 10x^2 + 2x^2 \cdot 7x + \ldots \\
= \ & \ldots + (\underbrace{1 \cdot 2}_{\text{Case 1}} + \underbrace{4 \cdot 10}_{\text{Case 2}} + \underbrace{2 \cdot 7}_{\text{Case 3}})x^3 + \ldots
\end{aligned}
$$

If C is a chessboard made up of pairwise disjoint subboards C_1, C_2, …, C_n, then

$$R(C, x) = R(C_1, x) \cdot R(C_2, x) \cdot \ \ldots \ \cdot R(C_n, x).$$

Let $r_k(C)$ denote the number of ways of placing k nontaking rooks on the chessboard C.

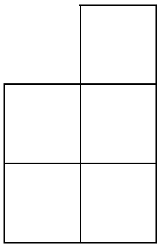
***C*:**



$r_k(C)$ consists of the following two parts.

1. A rook on the square designated by "*".

$\boldsymbol{C_s}$ **:**

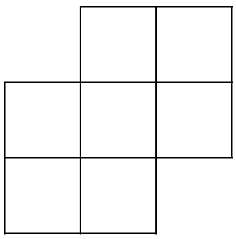


($\boldsymbol{C_s}$ is obtained from $\boldsymbol{C}$ by deleting the row and the column containing the designated square.)

$$\boldsymbol{r_{k-1}(C_s)} \textbf{ is included in } \boldsymbol{r_k(C)}.$$

2. No rook on the designated square.

$\boldsymbol{C_e}$ **:**



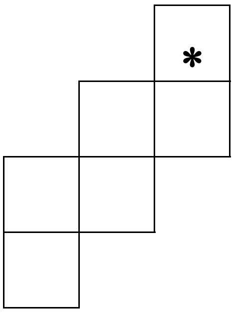
($\boldsymbol{C_e}$ is obtained from $\boldsymbol{C}$ by deleting the designated square.)

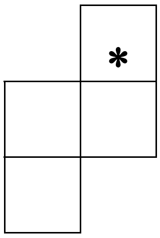
$$\boldsymbol{r_k(C_e)} \textbf{ is included in } \boldsymbol{r_k(C)}.$$

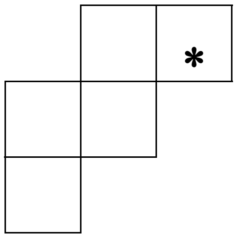
Therefore, $\boldsymbol{r_k(C) = r_{k-1}(C_s) + r_k(C_e)}$, **and**

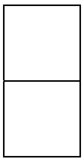
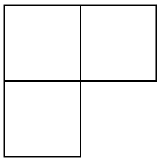
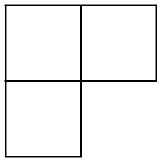
$$\boldsymbol{R(C, x) = xR(C_s, x) + R(C_e, x).}$$

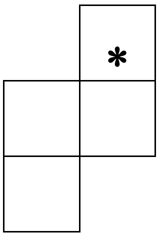
Ex.

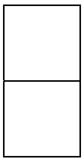
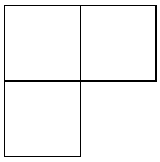
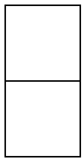
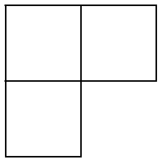


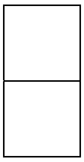
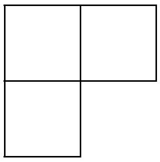
$= x \cdot$


$+$


$= x^2 \cdot$

$+ x \cdot$

$+ x \cdot$


$+$


$= x^2 \cdot$

$+ 2x \cdot$

$+ x \cdot$

$+$


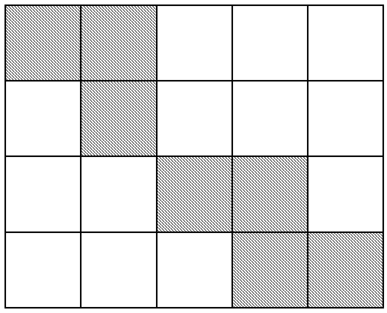
$= (x^2 + x) \cdot$

$+ (2x + 1) \cdot$


$$= (x^2 + x) \cdot (1 + 2x) + (2x + 1) \cdot (x^2 + 3x + 1)$$

$$= 1 + 6x + 10x^2 + 4x^3.$$

Ex.

***C* :**



Determine the number of ways of placing four nontaking rooks on the unshaded area of *C*.

Let c_i be the condition that a rook is placed on the shaded area of row *i*.

$$N(\overline{c}_1\,\overline{c}_2\,\overline{c}_3\,\overline{c}_4) = S_0 - S_1 + S_2 - S_3 + S_4$$

$$= 5! - r_1 \cdot 4! + r_2 \cdot 3! - r_3 \cdot 2! + r_4 \cdot 1!,$$

where R(the shaded area of C, x)

$$= (1 + 3x + x^2) \cdot (1 + 4x + 3x^2)$$
$$= 1 + 7x + 16x^2 + 13x^3 + 3x^4 = 1 + r_1 x + r_2 x^2 + r_3 x^3 + r_4 x^4.$$

$$\therefore\ N(\overline{c}_1\,\overline{c}_2\,\overline{c}_3\,\overline{c}_4) = 5! - 7 \cdot 4! + 16 \cdot 3! - 13 \cdot 2! + 3 \cdot 1!$$

$$= 25.$$

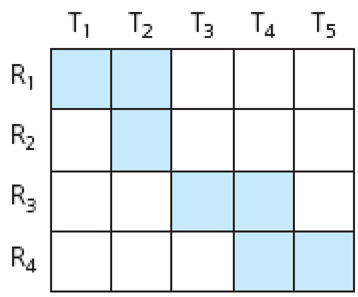
Ex. **Four people, denoted by R_1, R_2, R_3 and R_4, are assigned to five tables, denoted by T_1, T_2, T_3, T_4 and T_5, in a wedding reception. In how many ways can they be assigned to four distinct tables, subject to the following four restrictions:**

(a) R_1 is not assigned to T_1 or T_2;

(b) R_2 is not assigned to T_2;

(c) R_3 is not assigned to T_3 or T_4;

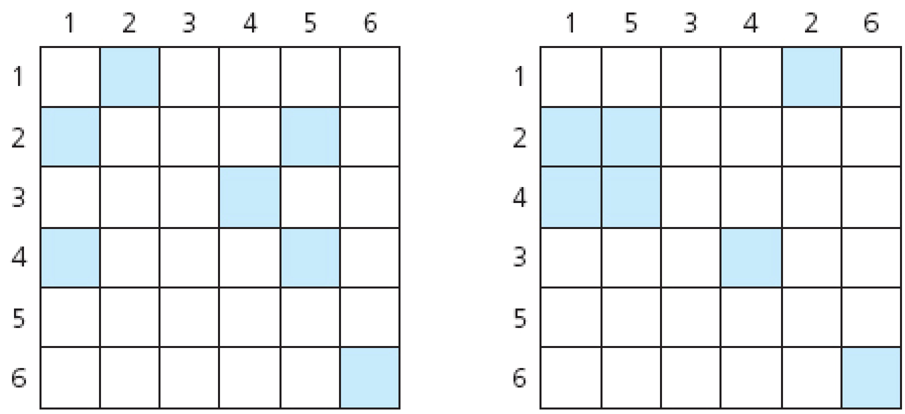
(d) R_4 is not assigned to T_4 or T_5.



The answer is 25, as computed in the example above.

Ex. **Two dice, denoted by *R* and *G*, are rolled six times. Under the condition of (*R*, *G*) $\notin$ {(1, 2), (2, 1), (2, 5), (3, 4), (4, 1), (4, 5), (6, 6)}, what is the probability that all six values 1, 2, …, 6 occur for both *R* and *G*?**

Consider the following left chessboard, where the row (column) labels represent the outcome on *R* (*G*).



The right chessboard is obtained by relabeling the rows and columns, where the seven shaded squares constitute four pairwise disjoint subboards.

Therefore, $r(C, x) = (1+4x+2x^2)\times(1+x)^3$

$= 1+7x+17x^2+19x^3+10x^4+2x^5.$

The probability is $P = \frac{6! \times S}{29^6}$, where there are 6! ways for 1, 2, 3, 4, 5, 6 to occur with *R* and *S* is the number of ways for 1, 2, 3, 4, 5, 6 to occur with *G* under the condition.

For example, (1, ?), (2, ?), (3, ?), (4, ?), (5, ?), (6, ?) and (4, ?), (5, ?), (6, ?), (1, ?), (2, ?), (3, ?) are two ways for 1, 2, 3, 4, 5, 6 to occur with *R*. Then, each way for the former to have 1, 2, 3, 4, 5, 6 occurring with *G*, e.g., (1, 6), (2, 2), (3, 3), (4, 4), (5, 5), (6, 1), uniquely corresponds to a way for the latter to have 1, 2, 3, 4, 5, 6 occurring with *G*, e.g., (4, 4), (5, 5), (6, 1), (1, 6), (2, 2), (3, 3), and *vice versa*.

Let c_i be the condition that when $R = i$, (R, G) is located at a shaded square.

$$\begin{aligned} S &= N(\overline{c}_1\,\overline{c}_2 \ldots \overline{c}_6) \\ &= 6! - 7 \times 5! + 17 \times 4! - 19 \times 3! + 10 \times 2! - 2 \times 1! \\ &= 192 \end{aligned}$$

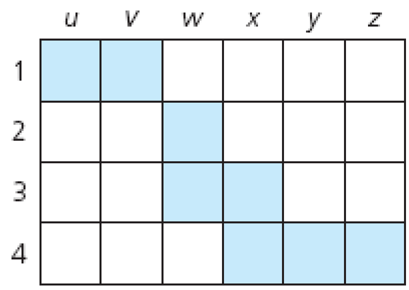
Thus, $P = \dfrac{6! \times 192}{29^6} \approx 0.00023.$

Ex. **How many one-to-one functions**

$$f : \{1, 2, 3, 4\} \rightarrow \{u, v, w, x, y, z\}$$

are there so that $f(1) \notin \{u, v\}$, $f(2) \notin \{w\}$, $f(3) \notin \{w, x\}$, **and** $f(4) \notin \{x, y, z\}$?

Consider the following chessboard.



$$r(C, x) = (1 + 2x) \times (1 + 6x + 9x^2 + 2x^3)$$

$$= 1 + 8x + 21x^2 + 20x^3 + 4x^4.$$

Let c_i **be the condition that** $(i, f(i))$ **is located at a shaded square.**

The answer is $N(\overline{c_1}\,\overline{c_2}\,\overline{c_3}\,\overline{c_4})$.

$$N(\overline{c}_1\,\overline{c}_2\,\overline{c}_3\,\overline{c}_4) = (6\times5\times4\times3) - 8\times(5\times4\times3) + 21\times(4\times3) - 20\times3 + 4\times1$$

$$= 76.$$

(do Exercise # 1)

Generating Functions

• Introductory Examples

Ex. There are three ways to choose one object from three distinct objects *a*, *b*, and *c* (namely, *a* or *b* or *c*). Similarly, there are three ways to choose two objects from them (namely, *ab* or *bc* or *ac*).

Interpretation by polynomials :

$+$: or

$\cdot$: and

x^i : to select *i* objects

$1+ax$: to select *a* or not to select *a*

$1+bx$: to select *b* or not to select *b*

$1+cx$: to select *c* or not to select *c*

$(1+ax)\cdot(1+bx)\cdot(1+cx)$: to select *a* or not to select *a*, to select *b* or not to select *b*, and to select *c* or not to select *c*.

$$(1+ax)\cdot(1+bx)\cdot(1+cx) = 1+(a+b+c)x+(ab+bc+ac)x^2+(abc)x^3$$

$$\Downarrow$$

The ways of selection are ϕ (to select none), *a* or *b* or *c* (to select one object), *ab* or *bc* or *ac* (to select two objects), *abc* (to select three objects).

Ex. Compute the number of solutions (x_1, x_2, x_3) for

$$x_1 + x_2 + x_3 = 12 \quad (4 \le x_1,\ 2 \le x_2,\ 2 \le x_3 \le 5).$$

$4 \le x_1 \le 8$: $(x^4 + x^5 + x^6 + x^7 + x^8)$

(The possible values for x_1 are 4, 5, 6, 7, and 8.)

$2 \le x_2 \le 6$: $(x^2 + x^3 + x^4 + x^5 + x^6)$

$2 \le x_3 \le 5$: $(x^2 + x^3 + x^4 + x^5)$

Each term $x^i x^j x^k$, where $i + j + k = 12$, in the product of $(x^4 + x^5 + x^6 + x^7 + x^8) \cdot (x^2 + x^3 + x^4 + x^5 + x^6) \cdot (x^2 + x^3 + x^4 + x^5)$ represents a solution, and *vice versa*.

So, the coefficient of x^{12} in the product is the number of solutions.

• Ordinary Generating Functions

Let $(a_0, a_1, a_2, \ldots, a_r, \ldots)$ be the symbolic representation of a sequence of events, or let it simply be a sequence of numbers. The function $F(x) = a_0\mu_0(x) + a_1\mu_1(x) + \ldots + a_r\mu_r(x) + \ldots$ is called the *ordinary generating function* of $(a_0, a_1, a_2, \ldots, a_r, \ldots)$, where $\mu_0(x), \mu_1(x), \ldots, \mu_r(x), \ldots$ is a sequence of functions of x that are used as indicators.

The indicator functions, $\mu_i(x)$, are usually chosen in such a way that no two distinct sequences will yield the same generating function.

Ex. $\mu_i(x)$: $1, 1+x, 1-x, 1+x^2, 1-x^2, \ldots, 1+x^r,$
$1-x^r, \ldots$

a_i : $3, 2, 6, 0, 0, \ldots$

$\Rightarrow \; F(x) = 3+2(1+x)+6(1-x) = 11-4x$

a_i : $1, 3, 7, 0, 0, \ldots$

$\Rightarrow \; F(x) = 1+3(1+x)+7(1-x) = 11-4x$

So, $1, 1+x, 1-x, 1+x^2, 1-x^2, \ldots$ **cannot be used as indicator functions.**

Ex. $\mu_i(x)$: $1, \cos x, \cos 2x, \ldots, \cos rx, \ldots$

a_i : $1, w, w^2, \ldots, w^r, \ldots$

$\Rightarrow \; F(x) = 1+w\cos x+w^2\cos 2x+\ldots+w^r\cos rx+\ldots$

The most usual and useful form of $\mu_i(x)$ is x^i :

$$F(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_i x^i + \ldots$$

for the sequence (a_0, a_1, a_2, …, a_i, …).

Ex. $(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots + \binom{n}{n}x^n$.

$(1+x)^n$ **is the generating function for** $\binom{n}{0}, \binom{n}{1},$

$\binom{n}{2}, \ldots, \binom{n}{n}, 0, 0, 0, \ldots$

Ex. $(1-x^{n+1}) = (1-x)(1+x+x^2+\ldots+x^n)$.

$(1-x^{n+1})/(1-x)$ **is the generating function for the**

sequence $\underbrace{1,\ 1,\ 1,\ \ldots,\ 1}_{n+1},\ 0,\ 0,\ 0,\ \ldots$

Ex. $\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots$

$$
\begin{array}{r l}
 & 1 + x + x^2 + \ldots \\
1 - x \,\big) & 1 \\
 & \underline{1 - x} \\
 & \quad x \\
 & \quad \underline{x - x^2} \\
 & \qquad x^2 \\
 & \qquad \underline{x^2 - x^3} \\
 & \qquad\quad x^3 \\
 & \qquad\quad \vdots
\end{array}
$$

$\dfrac{1}{1-x}$ **is the generating function for the sequence**

1, 1, 1, 1, …

$$\frac{d}{dx}\frac{1}{1-x} = \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \dots$$

$\frac{x}{(1-x)^2}$ **is the generating function for 0, 1, 2, 3, …**

$$\frac{d}{dx}\frac{x}{(1-x)^2} = \frac{1+x}{(1-x)^3} = 1 + 2^2x + 3^2x^2 + 4^2x^3 + \dots$$

$\frac{x(1+x)}{(1-x)^3}$ **is the generating function for** $0^2, 1^2, 2^2, 3^2, 4^2, \dots$

Let $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots$

$$f(x)/(1-x) = (a_0 + a_1x + a_2x^2 + a_3x^3 + \dots)(1 + x + x^2 + x^3 + \dots)$$

$$= (a_0) + (a_0 + a_1)x + (a_0 + a_1 + a_2)x^2 + \dots$$

$f(x)/(1-x)$ **is the generating function for** $a_0, a_0 + a_1,$

$a_0 + a_1 + a_2, \dots$

Ex. $\dfrac{x(1+x)}{(1-x)^3} = 0^2 + 1^2x + 2^2x^2 + 3^2x^3 + 4^2x^4 + \dots$

$\dfrac{x(1+x)}{(1-x)^4}$ **is the generating function for** $0^2, 0^2 + 1^2,$

$0^2 + 1^2 + 2^2, \dots$

When $n \notin Z^+$, $(1+x)^n = 1 + \sum_{r=1}^{\infty} \binom{n}{r} x^r$, where

$\binom{n}{r} = (n)(n-1)(n-2) \ldots (n-r+1)/r!$.

(derivable from Maclaurin series expansion)

Ex. $(1-x)^{-1}$

$$= 1 + \sum_{r=1}^{\infty} \binom{-1}{r}(-x)^r$$

$$= 1 + \sum_{r=1}^{\infty} [(-1)(-1-1)(-1-2) \ldots (-1-r+1)/r!](-1)^r x^r$$

$$= 1 + \sum_{r=1}^{\infty} [(-1)(-2)(-3) \ldots (-r)/r!](-1)^r x^r$$

$$= 1 + \sum_{r=1}^{\infty} x^r$$

$$= 1 + x + x^2 + x^3 + \ldots$$

Ex. Let $m \in Z^{+}$.

$$(1+x)^{-m}$$

$$= \quad 1+\sum_{r=1}^{\infty} \binom{-m}{r} x^r$$

$$= \quad 1+\sum_{r=1}^{\infty} ((-m)(-m-1)(-m-2) \ldots (-m-r+1)/r!)x^r$$

$$= \quad 1+\sum_{r=1}^{\infty} (-1)^r \binom{m+r-1}{r} x^r.$$

Ex. $(1+3x)^{-1/3} = 1 + \sum_{r=1}^{\infty} ((-1/3)(-4/3)(-7/3) \ldots$

$((-3r+2)/3)/r!)(3x)^r$

$= 1 + \sum_{r=1}^{\infty} ((-1)(-4)(-7) \ldots (-3r+2)/r!)x^r.$

Ex. Determine the coefficient of x^{15} in

$$F(x) = (x^2 + x^3 + x^4 + \ldots)^4.$$

$$\begin{aligned} F(x) &= (x^2(1 + x + x^2 + \ldots))^4 \\ &= x^8(1 + x + x^2 + \ldots)^4 \\ &= x^8(1/(1-x))^4 \\ &= x^8(1-x)^{-4}. \end{aligned}$$

The answer is the coefficient of x^7 in $(1-x)^{-4}$, which is

$$\binom{-4}{7}(-1)^7 = 120.$$

Ex. In how many ways can we select, with repetitions allowed, *r* objects from *n* distinct objects?

The problem is equivalent to finding the coefficient of x^r in

$$\begin{aligned}
F(x) &= (1+x+x^2+\ldots+x^r)^n = (1+x+x^2+\ldots+x^r+\ldots)^n \\
&= (1/(1-x))^n \\
&= (1-x)^{-n} \\
&= 1+\sum_{i=1}^{\infty}\binom{-n}{i}(-x)^i \\
&= 1+\sum_{i=1}^{\infty}\binom{n+(i-1)}{i}x^i,
\end{aligned}$$

which is $\binom{n+r-1}{r} = H\binom{n}{r}$.

Ex. Compute the number of integer solutions for

$$x_1 + x_2 + x_3 + x_4 = 24, \quad 3 \le x_i \le 8, \quad i = 1, 2, 3, 4.$$

The answer is the coefficient of x^{24} in

$$F(x) = (x^3 + x^4 + x^5 + x^6 + x^7 + x^8)^4$$

$$= x^{12}(1 + x + x^2 + x^3 + x^4 + x^5)^4$$

$$= x^{12}((1 - x^6)/(1 - x))^4$$

$$= x^{12}(1 - x^6)^4 (1 - x)^{-4}$$

$$= x^{12}(1 + \binom{4}{1}(-x^6) + \binom{4}{2}(-x^6)^2 + \binom{4}{3}(-x^6)^3 +$$

$$\binom{4}{4}(-x^6)^4)(1 + \binom{-4}{1}(-x) + \binom{-4}{2}(-x)^2 + \ldots),$$

which is $1 \times (-1)^{12}\binom{-4}{12} + \left((-1)\binom{4}{1}\right)\left((-1)^6\binom{-4}{6}\right) + \binom{4}{2} \times 1$

$$= 125.$$

Ex. Determine the coefficient of x^8 in

$$F(x) = \frac{1}{(x-3)(x-2)^2}.$$

$$F(x) = \frac{1}{(x-3)(x-2)^2} = \frac{1}{x-3} - \frac{1}{x-2} - \frac{1}{(x-2)^2}$$

$$= (\frac{-1}{3})(1-\frac{x}{3})^{-1} + (\frac{1}{2})(1-\frac{x}{2})^{-1} + (\frac{-1}{4})(1-\frac{x}{2})^{-2}$$

$$((1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots)$$

$$= (\frac{-1}{3})\sum_{i=0}^{\infty}(\frac{x}{3})^i + (\frac{1}{2})\sum_{i=0}^{\infty}(\frac{x}{2})^i +$$

$$(\frac{-1}{4})(1 + \binom{-2}{1}(-\frac{x}{2}) + \binom{-2}{2}(\frac{-x}{2})^2 + \ldots).$$

The coefficient of x^8 is $(\frac{-1}{3})(\frac{1}{3})^8 + (\frac{1}{2})(\frac{1}{2})^8 + (\frac{-1}{4})\binom{-2}{8}(\frac{-1}{2})^8.$

Ex. In how many ways can a police captain distribute 24 rifle shells to four police officers so that each gets at least three shells, but not more than eight?

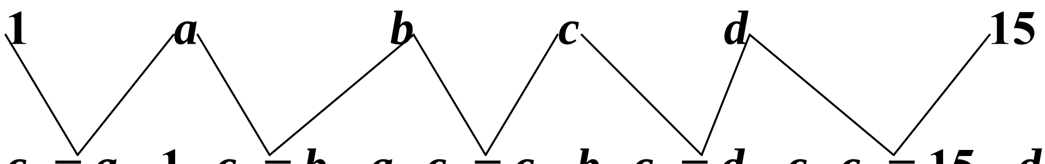
The problem is equivalent to finding the coefficient of x^{24} in $F(x)$

$$\begin{aligned} F(x) &= (x^3+x^4+x^5+x^6+x^7+x^8)^4 \\ &= x^{12}(1+x+x^2+x^3+x^4+x^5)^4, \end{aligned}$$

which is equal to 125, as computed in the example of page 59.

Ex. **Determine how many four-element subsets of $S = \{1, 2, \ldots, 15\}$ contain no consecutive integers.**

Let $\{a, b, c, d\}$ $(a < b < c < d)$ be an arbitrary four-element subset.



$c_1 = a - 1 \quad c_2 = b - a \quad c_3 = c - b \quad c_4 = d - c \quad c_5 = 15 - d$

$$c_1 + c_2 + c_3 + c_4 + c_5 = 14$$

set of four-element subsets ⟷ (**one-to-one correspondence**) **set of solutions to** $c_1 + c_2 + c_3 + c_4 + c_5 = 14$, $c_1, c_5 \geq 0$, $c_2, c_3, c_4 \geq 1$

set of required subsets ⟷ (**one-to-one correspondence**) **set of solutions to** $c_1 + c_2 + c_3 + c_4 + c_5 = 14$, $c_1, c_5 \geq 0$, $c_2, c_3, c_4 \geq 2$

The answer is the coefficient of x^{14} in

$$F(x) = (1+x+x^2+x^3+\ldots+x^8)^2\,(x^2+x^3+x^4+\ldots+x^{10})^3$$

$$= x^6(1+x+x^2+x^3+\ldots+x^8)^5$$

$$= x^6(1+x+x^2+x^3+\ldots+x^8+\ldots)^5$$

$$= x^6(1-x)^{-5},$$

which is $\binom{-5}{8}(-1)^8 = 495.$

• Partitions of Integers

Let $p(n)$ denote the number of partitions of a positive integer n into positive summands, disregarding their order.

$p(1) = 1$: 1.

$p(2) = 2$: 2, 1 + 1.

$p(3) = 3$: 3, 2 + 1, 1 + 1 + 1.

$p(4) = 5$: 4, 3 + 1, 2 + 2, 2 + 1 + 1, 1 + 1 + 1 + 1.

$p(10)$ is equal to the coefficient of x^{10} in

$$F(x) = (1 + x + x^2 + x^3 + \ldots) \times (1 + x^2 + x^4 + x^6 + \ldots) \times$$

$$(1 + x^3 + x^6 + x^9 + \ldots) \times \quad \ldots \quad \times (1 + x^{10} + x^{20} + \ldots)$$

$$= \frac{1}{1-x} \times \frac{1}{1-x^2} \times \frac{1}{1-x^3} \times \quad \ldots \quad \times \frac{1}{1-x^{10}}.$$

In general, $p(n)$ is equal to the coefficient of x^n in

$$F(x) = \prod_{i=1}^{\infty} \frac{1}{1-x^i}.$$

Although it is difficult to compute $p(n)$ from $F(x)$, it is still possible to compute the number of some restricted partitions on n by the aid of generating functions.

Ex. Find the generating function for the number of ways an advertising agent can purchase n minutes, if time slots of 30, 60, and 120 seconds are available.

Let 30 seconds be one time unit. Then the answer is the number of nonnegative integer solutions to

$$a + 2b + 4c = 2n,$$

(equivalently, the number of partitions of $2n$ into 1's, 2's and 4's), which is equal to the coefficient of x^{2n} in

$$\begin{aligned} F(x) &= (1 + x + x^2 + \ldots) \times (1 + x^2 + x^4 + \ldots) \times \\ &\quad (1 + x^4 + x^8 + \ldots) \\ &= \frac{1}{1-x} \times \frac{1}{1-x^2} \times \frac{1}{1-x^4}. \end{aligned}$$

Let $p_d(n)$ ($p_0(n)$) be the number of partitions of n whose summands are distinct (odd).

For example, there are five partitions of 4:

$$4,\ 3+1,\ 2+2,\ 2+1+1,\ 1+1+1+1,$$

where $p_d(4)=2$ (i.e., 4, $3+1$), and $p_0(4)=2$ (i.e., $3+1$ and $1+1+1+1$).

$p_d(n)$ is equal to the coefficient of x^n in

$$F(x) = (1+x)(1+x^2)(1+x^3)\ \ldots$$

$$= \prod_{i=1}^{\infty} (1+x^i),$$

where we define $p_d(0) = 1$.

$p_0(n)$ is equal to the coefficient of x^n in

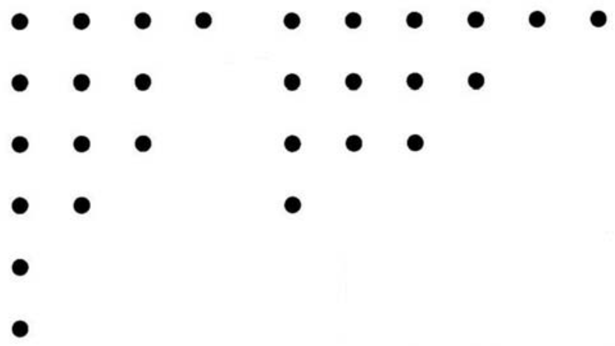
$$F(x) = (1+x+x^2+x^3+\ldots)\times(1+x^3+x^6+\ldots)\times$$

$$(1+x^5+x^{10}+\ldots)\times(1+x^7+x^{14}+\ldots)\times\ \ldots$$

$$= \frac{1}{1-x}\times\frac{1}{1-x^3}\times\frac{1}{1-x^5}\times\frac{1}{1-x^7}\times\ \ldots,$$

where we define $p_o(0) = 1$.

Ferrers graphs :   a dot representation of partitions



The number of dots per row in a Ferrers graph does not increase as we go from the top to the bottom.

The left Ferrers graph represents $14 = 4+3+3+2+1+1$, and the right Ferrers graph represents $14 = 6+4+3+1$; both are the transposition of each other, because one can be obtained from the other by interchanging rows and columns.

There is a one-to-one correspondence between a Ferrers graph and its transposition.

$\Rightarrow$ **the number of partitions of *n* into *m* summands is equal to the number of partitions of *n* whose maximal summand is *m*.**

($m = 4$ or 6 for the example above.)

• Exponential Generating Functions

$$F(x) = a_0\mu_0(x) + a_1\mu_1(x) + a_2\frac{\mu_2(x)}{2!} + a_3\frac{\mu_3(x)}{3!} + \ldots + a_r\frac{\mu_r(x)}{r!} + \ldots$$

is called the *exponential generating function* of the sequence (a_0, a_1, a_2, …, a_r, …), where $\mu_0(x)$, $\mu_1(x)$, $\mu_2(x)$, …, $\mu_r(x)$, … are the indicator functions.

When $\mu_i(x) = x^i$,

$$F(x) = a_0 + a_1x + a_2\frac{x^2}{2!} + a_3\frac{x^3}{3!} + \ldots + a_r\frac{x^r}{r!} + \ldots$$

Ex. $(1+x)^n = p(n, 0) + p(n, 1)x + p(n, 2)\frac{x^2}{2!} + p(n, 3)\frac{x^3}{3!} +$

$$\ldots + p(n, n)\frac{x^n}{n!}.$$

$(1+x)^n$ is the exponential generating function of

$p(n, 0), p(n, 1), p(n, 2), \ldots, p(n, n), \ldots$

Ex. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$

(derivable from Maclaurin series expansion)

e^x is the exponential generating function of

1, 1, 1, 1, 1, …

Ex. $$\frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots$$

$\frac{e^x + e^{-x}}{2}$ **is the exponential generating function of**

1, 0, 1, 0, 1, …

Ex. $$\frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots$$

$\frac{e^x - e^{-x}}{2}$ **is the exponential generating function of**

0, 1, 0, 1, 0, …

Let $x^p/p!$ indicate that there are p identical objects to be permuted.

Then, the number of ways to permute $p+q$ objects, where p of them is of one kind and q of them is of another kind, can be expressed as

$$\frac{x^p}{p!} \cdot \frac{x^q}{q!} = ((p+q)!/p!q!) \cdot \frac{x^{p+q}}{(p+q)!}$$

where the coefficient of $(p+q)!/p!q!$ is the answer.

Similarly, the number of ways to permute $p+q+r$ objects of three kinds can be expressed as

$$\frac{x^p}{p!} \cdot \frac{x^q}{q!} \cdot \frac{x^r}{r!} = ((p+q+r)!/p!q!r!) \cdot \frac{x^{p+q+r}}{(p+q+r)!}.$$

Ex. The number of ways to permute one, two, three, four, and five of five objects, with two of one kind and three of another kind, can be expressed as

$$(1+\frac{x}{1!}+\frac{x^2}{2!})\cdot(1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!})$$

(zero or one or two of the objects of the first kind) and (zero or one or two or three of the objects of the second kind) are permuted.

$$(1+\frac{x}{1!}+\frac{x^2}{2!})\cdot(1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!})$$

$$= 1+(\frac{1}{1!}+\frac{1}{1!})x+(\frac{1}{1!1!}+\frac{1}{2!}+\frac{1}{2!})x^2+(\frac{1}{1!2!}+\frac{1}{1!2!}+\frac{1}{3!})x^3+$$

$$(\frac{1}{1!3!}+\frac{1}{2!2!})x^4+(\frac{1}{2!3!})x^5$$

$$\left(\frac{1}{1!3!}+\frac{1}{2!2!}\right)\cdot x^4 = \left(\frac{4!}{1!3!}+\frac{4!}{2!2!}\right)\cdot \frac{x^4}{4!} = (4+6)\cdot \frac{x^4}{4!}.$$

$$\Downarrow$$

1. **There are 4 ways to permute four objects, with one of the first kind and three of the second kind.**
2. **There are 6 ways to permute four objects, with two of each kind.**
3. **There are $4+6$ ways to permute four of the five objects.**

Ex. The number of *r*-permutations of *n* distinct objects with unlimited repetitions is n^r.

$$(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^r}{r!})^n$$

$$= (1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots)^n$$

$$= e^{nx}$$

$$= \sum_{r=0}^{\infty} \frac{(nx)^r}{r!}$$

$$= \sum_{r=0}^{\infty} n^r \frac{x^r}{r!}.$$

Ex. In how many ways can four of the letters from ENGINE be arranged ?

selections of four letters	**the number of permutations**
EENN	**4!/2!2!**
EEGN	**4!/2!**
EEIN	**4!/2!**
EEGI	**4!/2!**
EGNN	**4!/2!**
EINN	**4!/2!**
GINN	**4!/2!**
EIGN	**4!**

(“E”, “N”)   (“G”, “I”)

$$F(x) = (1 + x + \frac{x^2}{2!})^2 \cdot (1 + x)^2.$$

The coefficient of $\frac{x^4}{4!}$ **in** $F(x)$ **is**

$$4!/2!2! + 4!/2! + 4!/2! + 4!/2! + 4!/2! + 4!/2! + 4!/2! + 4!.$$

Ex. A ship carries 48 flags, 12 each of red, white, blue and black. Twelve of these flags are placed on a vertical pole in order to communicate a signal to other ships. How many of these signals use an even number of blue and an odd number of black flags ?

$$F(x) = (1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ \dots)^2 \cdot (1+\frac{x^2}{2!}+\frac{x^4}{4!}+\ \dots) \cdot (x+\frac{x^3}{3!}+\frac{x^5}{5!}+\ \dots)$$

$$= (e^x)^2 \cdot (\frac{e^x+e^{-x}}{2}) \cdot (\frac{e^x-e^{-x}}{2})$$

$$= (\frac{1}{4}) \cdot (e^{2x}) \cdot (e^{2x}-e^{-2x})$$

$$= (\frac{1}{4}) \cdot (e^{4x}-1)$$

$$= \frac{1}{4}\ (\sum_{i=1}^{\infty} \frac{(4x)^i}{i!}).$$

The coefficient of $\frac{x^{12}}{12!}$ in $F(x)$ is 4^{11}.

Ex. A company hires 11 new employees. Each of these employees is to be assigned to one of four subdivisions with each subdivision getting at least one new employee. In how many ways can these assignments be made ?

E **:** $\{e_1, e_2, \ldots, e_{11}\}$

D **:** $\{d_1, d_2, d_3, d_4\}$

$f: \; E \to D$

How many onto functions *f* are there ?

$$F(x) = (x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots)^4 = (e^x - 1)^4$$

$$= e^{4x} - 4e^{3x} + 6e^{2x} - 4e^x + 1.$$

The coefficient of $\frac{x^{11}}{11!}$ **in** $F(x)$ **is**

$$4^{11} - 4 \cdot 3^{11} + 6 \cdot 2^{11} - 4 \cdot 1^{11}.$$

(This problem can be solved as well with the principle of inclusion and exclusion.)

Combinations $\rightarrow$ Ordinary Generation Functions

Permutations $\rightarrow$ Exponential Generation Functions

(do Exercise # 2)

Recurrence Relations

• Linear Recurrence Relations

Let $k \in Z^+$ and $c_n(\neq 0), c_{n-1}, c_{n-2}, \ldots, c_{n-k}(\neq 0)$ be constants.

If a_n is a discrete numeric function, then

$$c_n a_n + c_{n-1} a_{n-1} + c_{n-2} a_{n-2} + \ldots + c_{n-k} a_{n-k} = f(n), \quad n \geq k,$$

is *a linear recurrence relation with constant coefficients* of order k. The relation is *homogeneous* if $f(n) = 0$, and *nonhomogeneous* if $f(n) \neq 0$.

- **First-Order Linear Homogeneous Recurrence Relations with Constant Coefficients**

$a_n = ca_{n-1},\ \ n \geq 1,\ \ c$: constant.

$$
\begin{aligned}
a_n &= ca_{n-1} \\
&= c(ca_{n-2}) \\
&= c^2(ca_{n-3}) \\
&= c^3(ca_{n-4}) \\
&= c^4(ca_{n-5}) \\
&\ \ \vdots \\
&= c^{n-1}(ca_0) \\
&= c^n \cdot a_0.
\end{aligned}
$$

Ex. Solve $a_n = 7a_{n-1}$, $n \geq 1$, $a_2 = 98$.

$$\begin{aligned} a_n &= 7a_{n-1} \\ &= 7(7a_{n-2}) \\ &= 7^2(7a_{n-3}) \\ &\vdots \\ &= 7^{n-3}(7a_2) \\ &= 7^{n-2} \cdot 98 \\ &= 2 \cdot 7^n, \quad n \geq 0. \end{aligned}$$

Ex. Find a_{12} if $a_{n+1}^2 = 5a_n^2$, $a_n > 0$, $n \geq 0$, $a_0 = 2$.

Let $b_n = a_n{}^2$.

$b_{n+1} = 5b_n$, $n \geq 0$, $b_0 = 4$.

$b_n = 5^n \cdot b_0 = 4 \cdot 5^n$.

Therefore, $a_n = \sqrt{b_n} = 2 \cdot \left(\sqrt{5}\right)^n$, $n \geq 0$.

$a_{12} = 2 \cdot \left(\sqrt{5}\right)^{12} = 31250$.

• Second-Order Linear Homogeneous Recurrence Relations with Constant Coefficients

$$c_n a_n + c_{n-1} a_{n-1} + c_{n-2} a_{n-2} = 0, \quad n \geq 2 \qquad (1)$$

The solution of (1) has the form

$$a_n = c \cdot r^n, \qquad (2)$$

where $c \neq 0$ and $r \neq 0$.

Substituting (2) into (1), we have

$$c_n \cdot (c \cdot r^n) + c_{n-1} \cdot (c \cdot r^{n-1}) + c_{n-2} \cdot (c \cdot r^{n-2}) = 0.$$

$$\Rightarrow \quad c_n \cdot r^2 + c_{n-1} \cdot r + c_{n-2} = 0 \qquad (3)$$

(3) is called the *characteristic equation*, and its roots, denoted by r_1 and r_2, are called the *characteristic roots*.

Case 1. $r_1 \neq r_2$ **are real numbers.**

Ex. Solve $a_n + a_{n-1} - 6a_{n-2} = 0, \ \ n \geq 2, \ \ a_0 = 1, \ \ a_1 = 2.$

Let $a_n = c \cdot r^n$.

$\Rightarrow \ \ c \cdot r^n + c \cdot r^{n-1} - 6c \cdot r^{n-2} = 0$

characteristic equation : $r^2 + r - 6 = 0$

characteristic roots : $r_1 = 2, \ \ r_2 = -3$

$\Rightarrow \ \ a_n = c_1 \cdot 2^n + c_2 \cdot (-3)^n$ **is the general solution.**

c_1 **and** c_2 **can be determined by boundary conditions.**

$a_0 = 1 : \ \ c_1 + c_2 = 1$

$a_1 = 2 : \ \ 2c_1 - 3c_2 = 2$

$\Rightarrow \ \ c_1 = 1, \ \ c_2 = 0$

Therefore, $a_n = 2^n$ **is the unique solution.**

Ex. Solve the Fibonacci relation :

$$F_{n+2} = F_{n+1} + F_n,\ \ n \geq 0,\ \ F_0 = 0,\ \ F_1 = 1.$$

Let $F_n = c \cdot r^n$.

characteristic equation : $r^2 - r - 1 = 0$

characteristic roots : $r_1 = \frac{1+\sqrt{5}}{2},\ \ r_2 = \frac{1-\sqrt{5}}{2}$

general solution : $F_n = c_1 \cdot (\frac{1+\sqrt{5}}{2})^n + c_2 \cdot (\frac{1-\sqrt{5}}{2})^n$

$F_0 = 0$ **:** $c_1 + c_2 = 0$

$F_1 = 1$ **:** $c_1 \cdot \frac{1+\sqrt{5}}{2} + c_2 \cdot \frac{1-\sqrt{5}}{2} = 1$

$\Rightarrow\ c_1 = \frac{1}{\sqrt{5}},\ \ c_2 = \frac{-1}{\sqrt{5}}$

The unique solution is $F_n = \frac{1}{\sqrt{5}}((\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n)$.

Ex. For $n \geq 0$, let $S = \{1, 2, \ldots, n\}$ ($S = \varnothing$ if $n = 0$), and let a_n denote the number of subsets of S that contain no consecutive integers. Find and solve a recurrence relation for a_n .

The subsets of S that contain no consecutive integers are composed of two disjoint parts :

Part 1. those subsets containing n

a_{n-2} subsets are included in this part

Part 2. those subsets not containing n

a_{n-1} subsets are included in this part.

Thus, $a_n = a_{n-1} + a_{n-2}$ with $a_0 = 1$, $a_1 = 2$.

$$\Rightarrow \quad a_n = F_{n+2} = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n+2} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+2}\right)$$

$(F_0 = 0,\ F_1 = 1,\ F_2 = 1,\ F_3 = 2,\ \ldots$

$a_0 = 1,\ a_1 = 2,\ \ldots)$

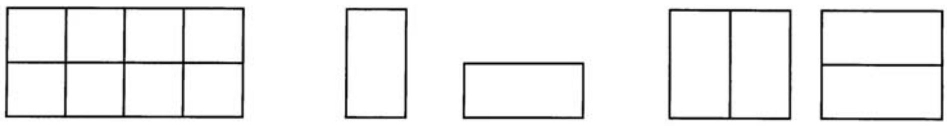
Ex. Find a recurrence relation for the number a_n of binary sequences of length n without consecutive 0's.

Let $a_n^{(0)}(a_n^{(1)})$: number of binary sequences of length n that contain no consecutive 0's and end with 0 (1).

$$a_n = a_n^{(0)} + a_n^{(1)} = a_{n-2} + a_{n-1}, \quad n \geq 2, \quad a_1 = 2, \quad a_2 = 3$$

$$\Rightarrow \quad a_n = F_{n+2} = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n+2} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+2}\right).$$

Ex. **In how many ways can a $2\times n$ chessboard (refer to the left graph for $n=4$) be covered by n 2×1 or 1×2 (refer to the middle graph) tiles?**



Let a_n be the answer, where $a_1=1$ and $a_2=2$ (refer to the right graph).

Considering the rightmost column of the chessboard, we have $a_n=a_{n-1}+a_{n-2}$ for $n\geq 3$.

$$\Rightarrow \quad a_n = F_{n+1} = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}-\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right).$$

Ex. In how many ways can n symbols from {0, 1, 2, …, 9} $\cup$ {+, $*$} form an arithmetic expression, e.g., 2+13$*$5? (a leading + is not allowed)

Let a_n be the answer.

$a_1 = 10$ (i.e., 0, 1, 2, …, 9).

$a_2 = 100$ (i.e., 00, 01, …, 09, 10, 11, …, 99).

Consider $n \geq 3$, and let xy be the rightmost two symbols, where $y \in \{0, 1, \ldots, 9\}$.

If $x \in \{0, 1, \ldots, 9\}$, then

$$a_n = 10 \times a_{n-1}.$$

If $x \in \{+, *\}$, then

$$a_n = 20 \times a_{n-2}.$$

Therefore, $a_n = 10a_{n-1} + 20a_{n-2}$.

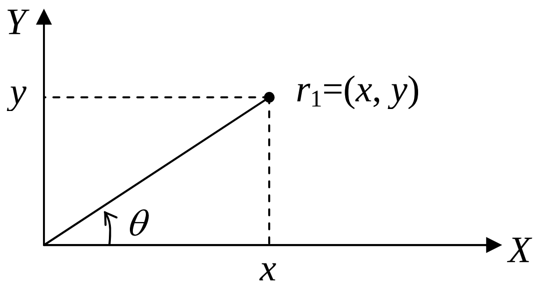
$$\Rightarrow \quad a_n = \frac{\sqrt{5}}{3}\left(\left(5+3\sqrt{5}\right)^n - \left(5-3\sqrt{5}\right)^n\right).$$

Case 2. r_1, r_2 **are complex numbers.**

$$r_1 = x + iy = \sqrt{x^2 + y^2}\,(\cos\theta + i\sin\theta)$$

$$r_2 = x - iy = \sqrt{x^2 + y^2}\,(\cos\theta - i\sin\theta)$$

$$\theta = \tan^{-1}\frac{y}{x}$$



If $z = r(\cos\theta \pm i\sin\theta)$**, then**

$z^n = r^n(\cos n\theta \pm i\sin n\theta)$.

Ex. $1+\sqrt{3}\,i = 2(\cos(\pi/3) + i\sin(\pi/3))$

$$
\begin{aligned}
(1+\sqrt{3}\,i)^{10} &= 2^{10}(\cos(10\pi/3) + i\sin(10\pi/3)) \\
&= 2^{10}(\cos(4\pi/3) + i\sin(4\pi/3)) \\
&= 2^{10}\left(\frac{-1}{2} - i\frac{\sqrt{3}}{2}\right) \\
&= (-2^{9})(1+\sqrt{3}\,i)
\end{aligned}
$$

Ex. Solve $a_n = 2(a_{n-1} - a_{n-2}),\ \ n \geq 2,\ \ a_0 = 1,\ \ a_1 = 2.$

Let $a_n = c \cdot r^n$.

$r^2 - 2r + 2 = 0$

$$\Rightarrow \quad r_1 = 1 + i = \sqrt{2}\,(\cos(\pi/4) + i\sin(\pi/4))$$

$$r_2 = 1 - i = \sqrt{2}\,(\cos(\pi/4) - i\sin(\pi/4))$$

general solution :

$$\begin{aligned} a_n &= c_1 \cdot r_1{}^n + c_2 \cdot r_2{}^n \\ &= c_1 \cdot (\sqrt{2}\,)^n(\cos(n\pi/4) + i\sin(n\pi/4)) + \\ &\quad\ c_2 \cdot (\sqrt{2}\,)^n(\cos(n\pi/4) - i\sin(n\pi/4)) \\ &= (\sqrt{2}\,)^n(k_1\cos(n\pi/4) + k_2\sin(n\pi/4)), \end{aligned}$$

where $k_1 = c_1 + c_2$ and $k_2 = (c_1 - c_2)i$.

$a_0 = 1 :\ \ k_1\cos 0 + k_2\sin 0 = 1$

$a_1 = 2 :\ \ \sqrt{2}\,(k_1\cos(\pi/4) + k_2\sin(\pi/4)) = 2$

$\Rightarrow\ \ k_1 = 1,\ \ k_2 = 1 \quad (c_1 = (1 - i)/2,\ \ c_2 = (1 + i)/2)$

$\Rightarrow\ \ a_n = (\sqrt{2}\,)^n(\cos(n\pi/4) + \sin(n\pi/4))$

Ex. Find the value of D_n, the $n \times n$ determinant given by

$$\begin{vmatrix} b & b & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ b & b & b & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & b & b & b & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b & b & b & \dots & 0 & 0 & 0 & 0 & 0 \\ . & . & . & . & . & \dots & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & \dots & b & b & b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & b & b & b & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & b & b & b \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & b & b \end{vmatrix}, \text{ where } b \in R^{+}.$$

Let a_n be the answer, where $a_1 = |b| = b$, $a_2 = \begin{vmatrix} b & b \\ b & b \end{vmatrix} = 0$,

and $a_3 = \begin{vmatrix} b & b & 0 \\ b & b & b \\ 0 & b & b \end{vmatrix} = -b^3$.

Expanding D_n by the first row, we have

$$a_n = b \times \begin{vmatrix} b & b & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ b & b & b & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & b & b & b & \dots & 0 & 0 & 0 & 0 \\ . & . & . & . & \dots & . & . & . & . \\ 0 & 0 & 0 & 0 & \dots & b & b & b & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & b & b & b \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & b & b \end{vmatrix} - b \times \begin{vmatrix} b & b & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & b & b & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & b & b & b & \dots & 0 & 0 & 0 & 0 \\ . & . & . & . & \dots & . & . & . & . \\ 0 & 0 & 0 & 0 & \dots & b & b & b & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & b & b & b \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & b & b \end{vmatrix}$$

$= b \times a_{n-1} - b^2 \times a_{n-2}$ **(expand the right determinant by the first column).**

$$\Rightarrow \quad a_n = b^n \times (\cos(n\pi/3) + \frac{1}{\sqrt{3}} \times \sin(n\pi/3)).$$

Case 3. $r_1 = r_2$ are real numbers.

Let r be the characteristic root (r is called *a root of multiplicity* 2).

general solution : $a_n = c_1 \cdot r^n + c_2 \cdot nr^n$

Ex. Solve $a_{n+2} = 4a_{n+1} - 4a_n, \quad n \ge 0, \quad a_0 = 1, \quad a_1 = 3.$

Let $a_n = c \cdot r^n$.

$r^2 - 4r + 4 = 0$

$\Rightarrow \quad r = 2$ (a root of multiplicity 2)

general solution : $a_n = c_1 \cdot 2^n + c_2 \cdot n2^n$

$a_0 = 1 : \quad c_1 = 1$

$a_1 = 3 : \quad 2c_1 + 2c_2 = 3$

$\Rightarrow \quad c_1 = 1, \quad c_2 = \frac{1}{2}$

$\Rightarrow \quad a_n = 2^n + n2^{n-1}$

• Linear Homogeneous Recurrence Relations of Higher Order with Constant Coefficients

Let $a_n = cr^n$.

1. **If all characteristic roots** $r_1, r_2, \ldots, r_k$ **are distinct, then the general solution has the following form**

$$a_n = c_1 r_1^{\,n} + c_2 r_2^{\,n} + \ldots + c_k r_k^{\,n}.$$

2. **If** r **is a characteristic root of multiplicity** m**, then the general solution includes the following as a component**

$$c_0 r^n + c_1 n r^n + c_2 n^2 r^n + \ldots + c_{m-1} n^{m-1} r^n.$$

Ex. Solve $2a_{n+3} = a_{n+2} + 2a_{n+1} - a_n,\ \ n \geq 0,\ \ a_0 = 0,$

$a_1 = 1,\ \ a_2 = 2.$

Let $a_n = c \cdot r^n.$

$$2r^3 - r^2 - 2r + 1 = 0$$

$$\Rightarrow\ r_1 = \frac{1}{2},\ \ r_2 = 1,\ \ r_3 = -1$$

general solution : $a_n = c_1(\frac{1}{2})^n + c_2 1^n + c_3(-1)^n$

$a_0 = 0:\ \ c_1 + c_2 + c_3 = 0$

$a_1 = 1:\ \ \frac{1}{2}c_1 + c_2 - c_3 = 1$

$a_2 = 2:\ \ \frac{1}{4}c_1 + c_2 + c_3 = 2$

$$\Rightarrow\ c_1 = \frac{-8}{3},\ \ c_2 = \frac{5}{2},\ \ c_3 = \frac{1}{6}$$

$$\Rightarrow\ a_n = (\frac{-8}{3})(\frac{1}{2})^n + (\frac{1}{6})(-1)^n + \frac{5}{2}$$

Ex. Solve $a_n + 6a_{n-1} + 12a_{n-2} + 8a_{n-3} = 0, \quad n \geq 3, \quad a_0 = 1,$

$a_1 = -2, \quad a_2 = 8.$

Let $a_n = c \cdot r^n$.

$r^3 + 6r^2 + 12r + 8 = 0$

$\Rightarrow$ $r = -2$ (a root of multiplicity 3)

general solution : $a_n = c_1(-2)^n + c_2 n(-2)^n + c_3 n^2(-2)^n$

$a_0 = 1: \quad c_1 = 1$

$a_1 = -2: \quad -2c_1 - 2c_2 - 2c_3 = -2$

$a_2 = 8: \quad 4c_1 + 8c_2 + 16c_3 = 8$

$\Rightarrow \quad c_1 = 1, \quad c_2 = \frac{-1}{2}, \quad c_3 = \frac{1}{2}$

$\Rightarrow \quad a_n = (-2)^n - \frac{1}{2} n(-2)^n + \frac{1}{2} n^2(-2)^n$

Ex. Solve $a_n - 4a_{n-1} + 5a_{n-2} - 2a_{n-3} = 0$, $n \geq 3$, $a_0 = 1$,

$a_1 = 3$, $a_2 = 6$.

Let $a_n = c \cdot r^n$.

$r^3 - 4r^2 + 5r - 2 = 0$

$\Rightarrow$ $r_1 = 1$ (a root of multiplicity 2)

$r_2 = 2$

general solution : $a_n = (c_1(1)^n + c_2 n(1)^n) + c_3(2)^n$

$a_0 = 1$: $c_1 + c_3 = 1$

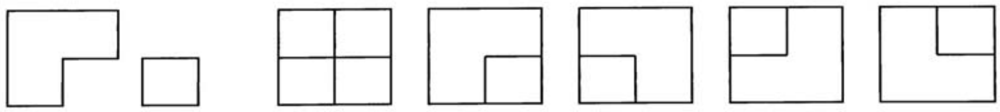
$a_1 = 3$: $c_1 + c_2 + 2c_3 = 3$

$a_2 = 6$: $c_1 + 2c_2 + 4c_3 = 6$

$\Rightarrow$ $c_1 = 0$, $c_2 = 1$, $c_3 = 1$

$\Rightarrow$ $a_n = n + 2^n$

Ex. In how many ways can a $2 \times n$ chessboard be covered by 1-square or 3-square tiles (refer to the left graph)?



Let a_n be the answer, where $a_1 = 1$, $a_2 = 5$ (refer to the right graph), and $a_3 = 11$.

Consider $n \geq 4$.

When the rightmost column is covered by two 1-square tiles, we have $a_n = a_{n-1}$.

When the rightmost column is covered by one 1-square tile and one 3-square tile, we have $a_n = 2a_{n-2}$.

When the rightmost column is covered by one 3-square tile, we have

$a_n = 2a_{n-2}$, **if the second rightmost column is covered by one 1-square tile and one 3-square tile;**

$a_n = 2a_{n-3}$, **if the second rightmost column is covered by two 3-square tiles.**

Therefore, $a_n = a_{n-1} + 4a_{n-2} + 2a_{n-3}$.

$$\Rightarrow \quad a_n = (-1)^n + \left(\frac{1}{\sqrt{3}}\right)\left(1+\sqrt{3}\right)^n + \left(\frac{-1}{\sqrt{3}}\right)\left(1-\sqrt{3}\right)^n.$$

(do Exercise #3)

- **Linear Nonhomogeneous Recurrence Relations**

$c_n a_n + c_{n-1} a_{n-1} + c_{n-2} a_{n-2} + \ldots + c_{n-k} a_{n-k} = f(n), \quad n \geq k,$

$f(n) \neq 0$, c_i $(n-k \leq i \leq n)$ **constant.**

$a_n = a_n^h + a_n^p$

a_n^h : **the homogeneous solution, which satisfies the equation with** $f(n) = 0$**, i.e.,**

$c_n a_n^h + c_{n-1} a_{n-1}^h + c_{n-2} a_{n-2}^h + \ldots + c_{n-k} a_{n-k}^h = 0$

a_n^p : **a particular solution to the equation, i.e.,**

$c_n a_n^p + c_{n-1} a_{n-1}^p + c_{n-2} a_{n-2}^p + \ldots + c_{n-k} a_{n-k}^p = f(n)$

There is no general way to find a_n^p.

However, when $f(n)$ is in a relatively simple form, a_n^p can be determined by method of undetermined coefficients (i.e., using the form of $f(n)$ to suggest a form for a_n^p).

Ex. Solve $a_n + 2a_{n-1} = n + 3, \quad a_0 = 3.$

Let $a_n^p = cn + d.$

$$(cn + d) + 2(c(n-1) + d) = n + 3$$

$$\Rightarrow \quad 3cn + (3d - 2c) = n + 3$$

$$\Rightarrow \quad 3c = 1, \quad 3d - 2c = 3$$

$$\Rightarrow \quad c = \frac{1}{3}, \quad d = \frac{11}{9}$$

$$a_n^p = \frac{1}{3}n + \frac{11}{9}$$

$$a_n^h = k(-2)^n$$

$$a_n = a_n^h + a_n^p = k(-2)^n + \frac{1}{3}n + \frac{11}{9}$$

$$a_0 = 3: \quad k + \frac{11}{9} = 3 \quad \Rightarrow \quad k = \frac{16}{9}$$

Ex. Solve $a_n - a_{n-1} = n-1,\ \ n \ge 1,\ \ a_1 = 0.$

First, try $a_n^p = cn + d.$

$$(cn+d)-(c(n-1)+d) = n-1$$

$\Rightarrow\ \ c = n-1,$ **a contradiction**

Then, try $a_n^p = cn^2 + dn + e.$

$$(cn^2+dn+e)-(c(n-1)^2+d(n-1)+e) = n-1$$

$$\Rightarrow\ \ 2cn - c + d = n-1$$

$$\Rightarrow\ \ c = \frac{1}{2},\ \ d = \frac{-1}{2}$$

$$a_n^p = \frac{1}{2}n^2 - \frac{1}{2}n + e = \frac{1}{2}n^2 - \frac{1}{2}n \quad (\text{let } e=0)$$

$$a_n^h = k(1)^n = k$$

$$a_n = a_n^h + a_n^p = k + \frac{1}{2}n^2 - \frac{1}{2}n$$

$$a_1 = 0:\ \ k + \frac{1}{2} - \frac{1}{2} = 0 \quad \Rightarrow \quad k = 0$$

another solution method :

$$
\begin{aligned}
a_n &= a_{n-1} + (n-1) \\
&= (a_{n-2} + (n-2)) + (n-1) \\
&= (a_{n-3} + (n-3)) + (n-2) + (n-1) \\
&\vdots \\
&= a_1 + 1 + 2 + \ \dots\ + (n-2) + (n-1) \\
&= \frac{1}{2} n(n-1)
\end{aligned}
$$

In general,

$$\begin{aligned} a_n &= a_{n-1} + f(n) \\ &= (a_{n-2} + f(n-1)) + f(n) \\ &= (a_{n-3} + f(n-2)) + f(n-1) + f(n) \\ &\vdots \\ &= a_0 + \sum_{i=1}^{n} f(i) \end{aligned}$$

For example, if $f(n) = 3n^2$,

$$\begin{aligned} a_n &= a_{n-1} + 3n^2 \\ &= a_0 + \sum_{i=1}^{n} 3i^2 \\ &= a_0 + \frac{3}{6}(n)(n+1)(2n+1) \end{aligned}$$

Ex. Solve $a_n - 3a_{n-1} = 5(7^n), \quad n \geq 1, \quad a_0 = 2.$

$a_n^h = c(3^n).$

Let $a_n^p = k \cdot 7^n.$

$$k \cdot 7^n - 3k7^{n-1} = 5 \cdot 7^n \ \Rightarrow \ k = \frac{35}{4}$$

$$a_n = a_n^h + a_n^p = c(3^n) + \frac{35}{4} \cdot 7^n$$

$$a_0 = 2 \quad \Rightarrow \quad c = \frac{-27}{4}$$

Ex. Solve $a_n - 3a_{n-1} = 5(3^n)$, $n \geq 1$, $a_0 = 2$.

$a_n^h = c(3^n)$

Let $a_n^p = k \cdot 3^n$.

$k \cdot 3^n - 3k3^{n-1} = 5(3^n) \quad \Rightarrow \quad 0 = 5(3^n)$, **a contradiction!**

Let $a_n^p = kn3^n$.

$kn3^n - 3k(n-1)3^{n-1} = 5(3^n) \;\Rightarrow\; k = 5$

$a_n = a_n^h + a_n^p = c \cdot 3^n + 5n3^n$

$a_0 = 2 \;\Rightarrow\; c = 2$

1. First-order relations :

If $f(n) = q \cdot r^n$, q, r : **constants,**

$$a_n^p = \begin{cases} k \cdot r^n & \text{if } a_n^h \neq c \cdot r^n \\ k \cdot nr^n & \text{if } a_n^h = c \cdot r^n \end{cases}$$

2. Second-order relations :

If $f(n) = q \cdot r^n$, q, r : **constants,**

$$a_n^p = \begin{cases} k \cdot nr^n & \text{if } a_n^h = c_1 r^n + c_2 r_1{}^n \quad (r \neq r_1) \\ k \cdot n^2 r^n & \text{if } a_n^h = c_1 r^n + c_2 n r^n \\ k \cdot r^n & \text{else} \end{cases}$$

Ex. Solve $a_n - 4a_{n-1} + 4a_{n-2} = 2^n, \quad n \geq 2, \quad a_0 = 1, \quad a_1 = 2.$

$$a_n^h = c_1 2^n + c_2 n 2^n$$

Let $a_n^p = kn^2 2^n$.

$$kn^2 2^n - 4k(n-1)^2 2^{n-1} + 4k(n-2)^2 2^{n-2} = 2^n$$

$$\Rightarrow \quad k = \frac{1}{2}$$

$$a_n = a_n^h + a_n^p = c_1 2^n + c_2 n 2^n + \frac{1}{2} \cdot n^2 2^n$$

$$a_0 = 1, \quad a_1 = 2 \quad \Rightarrow \quad c_1 = 1, \quad c_2 = \frac{-1}{2}$$

Ex. Solve $a_{n+2} - 4a_{n+1} + 3a_n = -200$, $n \geq 0$, $a_0 = 3000$, $a_1 = 3300$.

$a_n^h = c_1 3^n + c_2 1^n = c_1 3^n + c_2$.

If we let $a_n^p = k$, then

$$k - 4k + 3k = -200, \quad \text{a contradiction!}$$

So, we let $a_n^p = kn$, and

$$k(n+2) - 4k(n+1) + 3kn = -200.$$

$$\Rightarrow \; k = 100.$$

Hence, $a_n = a_n^h + a_n^p = c_1 3^n + c_2 + 100n$.

$a_0 = 3000$, $a_1 = 3300 \;\Rightarrow\; c_1 = 100$, $c_2 = 2900$

Summary :

$c_n a_n + c_{n-1} a_{n-1} + \ldots + c_{n-k} a_{n-k} = f(n)$,

where $c_n, c_{n-1}, \ldots, c_{n-k}$ are constants.

$g(n)$	$h(g(n))$
c: constant	c_0: constant
n^t, $t \in z^+$	$c_0 + c_1 n + \ldots + c_{t-1} n^{t-1} + c_t n^t$
r^n, $r \in R$	$c_1 r^n$
$\sin \alpha n$ (α: constant)	$c_1 \sin \alpha n + c_2 \cos \alpha n$
$\cos \alpha n$	$c_1 \sin \alpha n + c_2 \cos \alpha n$
$n^t r^n$	$r^n (c_0 + c_1 n + \ldots + c_{t-1} n^{t-1} + c_t n^t)$
$r^n \sin \alpha n$	$r^n (c_1 \sin \alpha n + c_2 \cos \alpha n)$
$r^n \cos \alpha n$	$r^n (c_1 \sin \alpha n + c_2 \cos \alpha n)$

Case 1. $f(n) = q \cdot g(n)$, q **: constant.**

$$a_n^p = h(g(n)) \quad \text{if } g(n) \text{ is not included in } a_n^h;$$

$$n^s \cdot h(g(n)) \quad \text{else,}$$

where s **is the smallest integer so that** $n^s \cdot g(n)$ **is not included in** a_n^h**.**

Case 2. $f(n) = q_1 \cdot g_1(n) + q_2 \cdot g_2(n) + \ldots + q_k \cdot g_k(n)$,

where each q_i is a constant and each $g_i(n)$ is in the class of $g(n)$ $(1 \le i \le k)$.

$a_n^p = h_1(n) + h_2(n) + \ldots + h_k(n)$ **and**

$$h_i(n) = \begin{cases} h(g_i(n)) & \text{if } g_i(n) \text{ is not included in } a_n^h; \\ n^s \cdot h(g_i(n)) & \text{else,} \end{cases}$$

where s is the smallest integer so that $n^s \cdot g_i(n)$ is not included in a_n^h.

For example, if $f(n) = 4n^2 + 3\sin 2n$ and $a_n^h = c_0 n + c_1 2^n$,

$$a_n^p = (c_2 + c_3 n + c_4 n^2) + (c_5 \sin 2n + c_6 \cos 2n).$$

If $f(n) = 4n^2 + 2^n$ and $a_n^h = c_0 n^2 + c_1 2^n + c_2 n 2^n$, then

$$a_n^p = (c_3 + c_4 n + c_5 n^2 + c_6 n^3) + c_7 n^2 2^n.$$

Ex. Consider the following recurrence relation :

$$a_{n+2} - 10a_{n+1} + 21a_n = f(n), \quad n \ge 0.$$

$$a_n^h = c_1 3^n + c_2 7^n.$$

$f(n)$	a_n^p
5	k
$3n^2 - 2$	$k_2 n^2 + k_1 n + k_0$
$7(11^n)$	$k(11^n)$
$6(3^n)$	$kn(3^n)$
$2(3^n) - 8(9^n)$	$k_1 n(3^n) + k_0(9^n)$
$4(3^n) + 3(7^n)$	$k_1 n(3^n) + k_0 n(7^n)$

In the above, k, k_0, k_1 and k_2 are all constants.

Ex. Consider the following recurrence relation :

$$a_n + 4a_{n-1} + 4a_{n-2} = f(n), \quad n \geq 2.$$

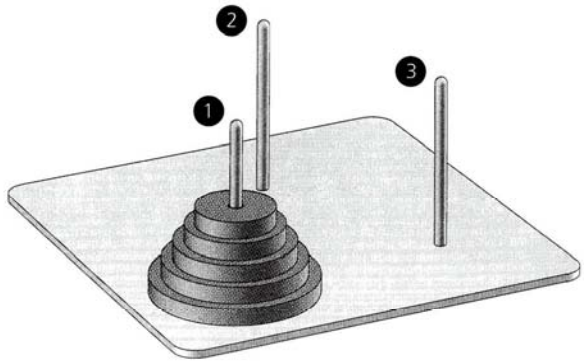
$$a_n^h = c_1(-2)^n + c_2 n(-2)^n.$$

$f(n)$	a_n^p
$5(-2)^n$	$kn^2(-2)^n$
$7n(-2)^n$	$n^2(k_1 n + k_0)(-2)^n$
$-11n^2(-2)^n$	$n^2(k_2 n^2 + k_1 n + k_0)(-2)^n$

In the above, k, k_0, k_1 and k_2 are all constants.

When $f(n) = 7n(-2)^n$, n induces $k_1 n + k_0$ in a_n^p, and the leading n^2 in a_n^p is to avoid $(-2)^n$ and $n(-2)^n$.

Ex. The Towers of Hanoi.



Let a_n be the minimal number of disk moves required to transfer n disks from peg 1 to peg 3.

($a_0 = 0$, $a_1 = 1$, $a_2 = 3$)

The *n* disks can be transferred as follows.

1. **Transfer top $n-1$ disks from peg 1 to peg 2.**
2. **Transfer the largest disk from peg 1 to peg 3.**
3. **Transfer the $n-1$ disks from peg 2 to peg 3.**

$$\Rightarrow \quad a_n \le 2a_{n-1} + 1.$$

On the other hand, the transfer of the largest disk takes at least one disk move and induces at least two transfers of $n-1$ disks.

$$\Rightarrow \quad a_n \ge 2a_{n-1} + 1.$$

Therefore, $a_n = 2a_{n-1} + 1.$

$$\Rightarrow \quad a_n = 2^n - 1, \ \ n \ge 0.$$

Ex. **Pauline takes out a loan of S dollars that is to be paid back in T periods of time. If r is the interest rate per period for the loan, what (constant) payment P must she make at the end of each period?**

Let a_n be the amount still owed at the end of the nth period. ($a_0 = S$ and $a_T = 0$)

When $n \geq 1$, $a_n = a_{n-1} + ra_{n-1} - P$.

$\Rightarrow \quad a_n = (S - P/r)(1 + r)^n + P/r$, **for** $0 \leq n \leq T$.

Since $a_T = 0$, we have $P = Sr(1 - (1 + r)^{-T})^{-1}$.

Ex. Find the number of comparisons needed, if a divide & conquer method is used to determine the maximal and minimal numbers of 2^n real numbers?

Let a_n be the answer, where $a_1 = 1$.

When $n > 1$, $a_n = 2a_{n-1} + 2$.

$\Rightarrow \quad a_n = (3/2)2^n - 2.$

Ex. Find the number a_n of quaternary sequences of length n having an even number of 1's.

$a_1 = 3$. Consider $n \geq 2$ below.

When the rightmost digit is 0 or 2 or 3,

$$a_n = a_{n-1}.$$

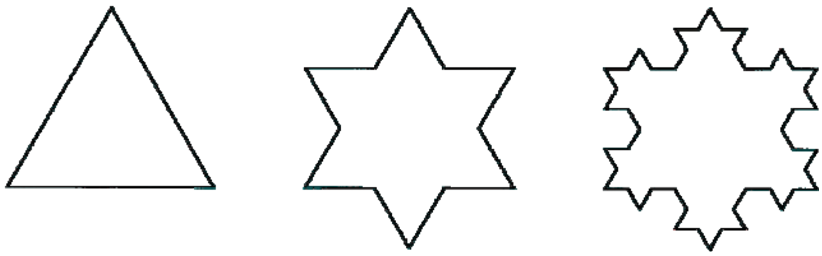
When the rightmost digit is 1, the other $n-1$ digits should have an odd number of 1's, i.e.,

$$a_n = 4^{n-1} - a_{n-1}.$$

Therefore, $a_n = 3a_{n-1} + (4^{n-1} - a_{n-1}) = 2a_{n-1} + 4^{n-1}$.

$$\Rightarrow \quad a_n = 2^{n-1} + 2(4^{n-1}).$$

Ex.



P_0 $\quad$ P_1 $\quad$ P_2

P_0 : an equilateral triangle of side length 1.

P_1 : a polygon obtained by replacing the middle one-third of each side of P_0 with a new equilateral triangle of side length 1/3.

P_2 : a polygon obtained from P_1 in a similar way.

An equilateral triangle of side length s has area $(\sqrt{3}/4)s^2$.

Let a_n be the area of P_n, where

$$a_0 = \sqrt{3}/4;$$

$$a_1 = \sqrt{3}/4 + 3 \times (\sqrt{3}/4) \times (1/3)^2 = \sqrt{3}/3;$$

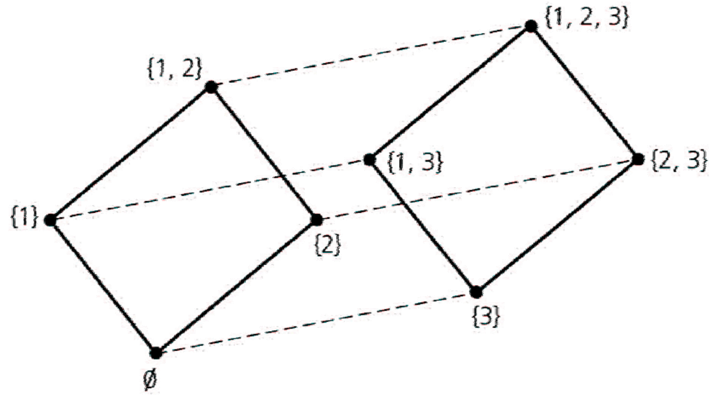
$$a_2 = \sqrt{3}/3 + 3 \times 4 \times (\sqrt{3}/4) \times (1/3^2)^2 = 10\sqrt{3}/27.$$

(P_n has 3×4^n sides each of length $1/3^n$.)

$$\begin{aligned} \Rightarrow \quad a_n &= a_{n-1} + 3 \times 4^{n-1} \times (\sqrt{3}/4) \times (1/3^n)^2 \\ &= a_{n-1} + 1/(4\sqrt{3}) \times (4/9)^{n-1} \\ &= (1/(5\sqrt{3}))(6 - (4/9)^{n-1}), \quad n \geq 0. \end{aligned}$$

Ex. **Let $S_n = \{1, 2, \ldots, n\}$ and $\wp(S_n)$ denote the power set of S_n. Find the number a_n of edges in the Hasse diagram for the partial order $(\wp(S_n), \subseteq)$.**

$n = 3$



$a_1 = 1$, $a_2 = 4$, **and** $a_3 = 12$ $(= 2a_2 + 2^2)$.

The Hasse diagram for $(\wp(S_n), \subseteq)$ contains two Hasse diagrams, one for $(\wp(S_{n-1}), \subseteq)$ and the other for $(\{\{n\} \cup T : T \in \wp(S_{n-1})\}, \subseteq)$, which are joined with 2^{n-1} edges.

Therefore, $a_n = 2a_{n-1} + 2^{n-1}$.

$\Rightarrow \quad a_n = n2^{n-1}, \ n \geq 1.$

• Method of Generating Functions

$$c_0 a_n + c_1 a_{n-1} + \ldots + c_r a_{n-r} = f(n) \qquad (1)$$

Assume that (1) is valid for $n \geq k$.

Let $A(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$ denote the ordinary generating function of the sequence $(a_0, a_1, a_2, \ldots, a_n, \ldots)$.

1. Multiply both sides of (1) by x^n.

$$(c_0 a_n + c_1 a_{n-1} + \ldots + c_r a_{n-r}) x^n = f(n) x^n \qquad (2)$$

2. Sum both sides of (2) from $n = k$ to ∞.

$$\sum_{n=k}^{\infty} (c_0 a_n + c_1 a_{n-1} + \ldots + c_r a_{n-r}) x^n = \sum_{n=k}^{\infty} f(n) x^n$$

$$c_0 \sum_{n=k}^{\infty} a_n x^n + c_1 x \sum_{n=k}^{\infty} a_{n-1} x^{n-1} + \ldots + c_r x^r \sum_{n=k}^{\infty} a_{n-r} x^{n-r} = \sum_{n=k}^{\infty} f(n) x^n$$

$$c_0(A(x) - a_0 - a_1x - \ldots - a_{k-1}x^{k-1}) \quad +$$

$$c_1x(A(x) - a_0 - a_1x - \ldots - a_{k-2}x^{k-2}) \quad + \quad \ldots \quad +$$

$$c_rx^r(A(x) - a_0 - a_1x - \ldots - a_{k-r-1}x^{k-r-1})$$

$$= \sum_{n=k}^{\infty} f(n)x^n \qquad (3)$$

3. Solve (3) for $A(x)$.

4. Determine the coefficient a_n of x^n in $A(x)$.

Ex. Solve $a_n - 3a_{n-1} = n, \quad n \geq 1, \quad a_0 = 1.$

1. $(a_n - 3a_{n-1})x^n = \; nx^n$

2. $\sum_{n=1}^{\infty} (a_n - 3a_{n-1})x^n \; = \; \sum_{n=1}^{\infty} nx^n$

$$\sum_{n=1}^{\infty} a_n x^n \; - \; 3\sum_{n=1}^{\infty} a_{n-1}x^n \; = \; \sum_{n=1}^{\infty} nx^n$$

$$(A(x) - a_0) - 3xA(x) = \frac{x}{(1-x)^2}$$

$$\left(\frac{1}{(1-x)^2} = \frac{d}{dx}\frac{1}{1-x} = 1 + 2x + 3x^2 + \; \ldots\right)$$

3. $A(x) \; = \; \frac{1}{1-3x} + \frac{x}{(1-x)^2(1-3x)}$

$$= \frac{7}{4(1-3x)} + \frac{-1}{4(1-x)} + \frac{-1}{2(1-x)^2}$$

$$\left(\frac{1}{1-3x} = 1 + 3x + (3x)^2 + (3x)^3 + \; \ldots\right)$$

4. $a_n = \frac{7}{4}3^n - \frac{n}{2} - \frac{3}{4}, \quad n \geq 0$

Ex. Solve $a_n - 2a_{n-1} = 4^{n-1}, \quad n \geq 1, \quad a_0 = 1.$

1. $(a_n - 2a_{n-1})x^n = 4^{n-1}x^n$

2. $$\sum_{n=1}^{\infty} (a_n - 2a_{n-1})x^n = \sum_{n=1}^{\infty} 4^{n-1}x^n$$

$$\sum_{n=1}^{\infty} a_n x^n - 2\sum_{n=1}^{\infty} a_{n-1}x^n = \sum_{n=1}^{\infty} 4^{n-1}x^n$$

$$(A(x) - 1) - 2xA(x) = \frac{x}{1-4x}$$

3. $$A(x) = \frac{1}{2(1-4x)} + \frac{1}{2(1-2x)}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty} (4x)^n + \frac{1}{2}\sum_{n=0}^{\infty} (2x)^n$$

4. $a_n = \frac{1}{2}4^n + \frac{1}{2}2^n = \frac{1}{2}4^n + 2^{n-1}, \quad n \geq 0$

Ex. **Solve** $a_{n+2} - 5a_{n+1} + 6a_n = 2,\ \ n \geq 0,\ \ a_0 = 3,\ \ a_1 = 7.$

1. $(a_{n+2} - 5a_{n+1} + 6a_n)x^{n+2} = 2x^{n+2}.$

2. $\sum_{n=0}^{\infty} (a_{n+2} - 5a_{n+1} + 6a_n)x^{n+2} = \sum_{n=0}^{\infty} 2x^{n+2}.$

$$\sum_{n=0}^{\infty} a_{n+2}x^{n+2} - 5\sum_{n=0}^{\infty} a_{n+1}x^{n+2} + 6\sum_{n=0}^{\infty} a_n x^{n+2}$$

$$= 2\sum_{n=0}^{\infty} x^{n+2}.$$

$$(A(x) - a_0 - a_1x) - 5x(A(x) - a_0) + 6x^2A(x) = \frac{2x^2}{1-x}.$$

3. $A(x) = \dfrac{2}{1-3x} + \dfrac{1}{1-x} = 2\sum_{n=0}^{\infty} (3x)^n + \sum_{n=0}^{\infty} x^n.$

4. $a_n = 2(3)^n + 1,\ \ n \geq 0.$

• Nonlinear Recurrence Relations

There is no general method to solve nonlinear recurrence relations.

Type 1. $b_{n+1} = b_0 b_n + b_1 b_{n-1} + \dots + b_{n-1} b_1 + b_n b_0 \quad (n \geq 0)$.

Let $B(x) = \sum_{n=0}^{\infty} b_n x^n$

be the generating function for $(b_0, b_1, \dots, b_n, \dots)$.

$$\sum_{n=0}^{\infty} b_{n+1} x^{n+1} = \sum_{n=0}^{\infty} (b_0 b_n + b_1 b_{n-1} + \dots + b_{n-1} b_1 + b_n b_0) x^{n+1}$$

$$B(x) - b_0 = xB^2(x).$$

$$\Rightarrow \quad B(x) = \frac{1}{2x}(1 \pm \sqrt{1-4b_0 x})$$

$$= \frac{1}{2x}\left(1 \pm \sum_{n=0}^{\infty} \frac{-b_0^n}{2n-1}\binom{2n}{n}x^n\right)$$

(refer to page 489 of Grimaldi's book)

Since $b_n \geq 0$,

$$B(x) = \frac{1}{2x}\left(1 - \sum_{n=0}^{\infty} \frac{-b_0^n}{2n-1}\binom{2n}{n}x^n\right)$$

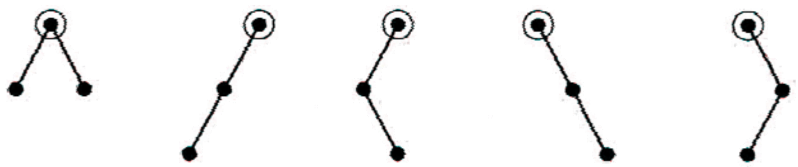
$$= \frac{1}{2x}\sum_{n=1}^{\infty} \frac{b_0^n}{2n-1}\binom{2n}{n}x^n$$

$$\Rightarrow \quad b_n = \frac{1}{2}\left(\frac{b_0^{n+1}}{2(n+1)-1}\right)\binom{2(n+1)}{n+1}$$

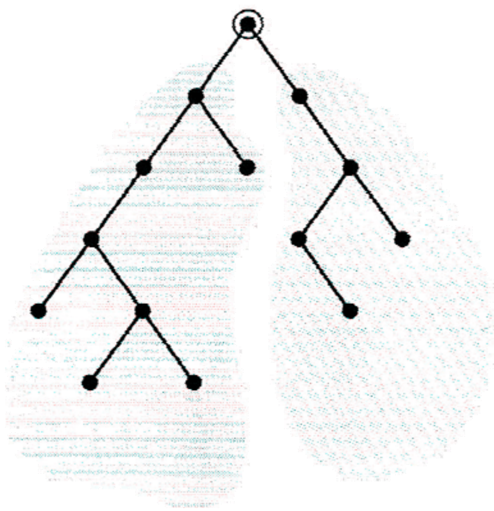
$$= \frac{b_0^{n+1}}{n+1}\binom{2n}{n}$$

Ex. **Find the number b_n of ordered rooted binary trees on n vertices.**

$$b_0 = 1, \quad b_1 = 1, \quad b_2 = 2, \quad b_3 = 5.$$



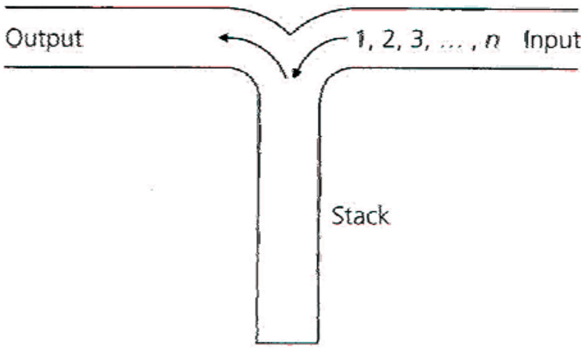
$$b_n = b_0 b_{n-1} + b_1 b_{n-2} + \ \ldots \ + b_{n-2} b_1 + b_{n-1} b_0.$$



one instance of

$b_8 b_5$

Ex. Find the number of distinct outputs that may be generated from the following stack.



Each output of the stack is a permutation of 1, 2, …, *n*.

When $n = 4$, there are 14 permutations that may be generated by the stack.

1, 2, 3, 4	**2, 1, 3, 4**	**2, 3, 1, 4**	**2, 3, 4, 1**
1, 2, 4, 3	**2, 1, 4, 3**	**3, 2, 1, 4**	**2, 4, 3, 1**
1, 3, 2, 4			**3, 2, 4, 1**
1, 3, 4, 2			**3, 4, 2, 1**
1, 4, 3, 2			**4, 3, 2, 1**

Suppose that $x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n$ is a permutation generated by the stack, where $1 \le i \le n$.

Then,

$$\{x_1, \ldots, x_{i-1}\} = \{2, \ldots, i\} \text{ and } \{x_{i+1}, \ldots, x_n\} = \{i+1, \ldots, n\}.$$

Let b_i be the number of distinct outputs that may be generated from the stack with i consecutive integers as inputs.

$$\Rightarrow \quad b_n = b_0 b_{n-1} + b_1 b_{n-2} + \ \ldots \ + b_{n-2} b_1 + b_{n-1} b_0$$

For example, 2, 1, 4, 3 is an instance of $b_1 b_2$ and 3, 4, 2, 1 is an instance of $b_3 b_0$.

Type 2. $b_{n+1} = a_0 b_n + a_1 b_{n-1} + \ \dots \ + a_{n-1} b_1 + a_n b_0 \quad (n \geq 0)$.

Let $A(x) = \sum_{n=0}^{\infty} a_n x^n$ **and** $B(x) = \sum_{n=0}^{\infty} b_n x^n$

be the generating functions for $(a_0, a_1, \dots, a_n, \dots)$

and $(b_0, b_1, \dots, b_n, \dots)$**, respectively.**

$$\sum_{n=0}^{\infty} b_{n+1} x^{n+1} = \sum_{n=0}^{\infty} (a_0 b_n + a_1 b_{n-1} + \ \dots \ + a_{n-1} b_1 + a_n b_0) x^{n+1}$$

$$B(x) - b_0 = xA(x)B(x).$$

If either $A(x)$ **or** $B(x)$ **is known, then the other can be obtained.**

• Recurrence Relations with Two Indices

Ex. Find $a(n, r)$, the number of ways we can select, with repetition, r objects from n distinct objects b_1, b_2, …, b_n.

Consider object b_1.

1. **$a(n-1, r)$ of $a(n, r)$ ways do not select b_1.**
2. **$a(n, r-1)$ of $a(n, r)$ ways select b_1 at least once.**

$\Rightarrow$ $a(n, r) = a(n-1, r) + a(n, r-1)$

Let $f_n(x) = \sum_{r=0}^{\infty} a(n, r)x^r$ **be the generating function for** $(a(n, 0), a(n, 1), a(n, 2), \ldots)$.

$$\sum_{r=1}^{\infty} a(n, r)x^r = \sum_{r=1}^{\infty} (a(n-1, r) + a(n, r-1))x^r$$

$$f_n(x) - a(n, 0) = (f_{n-1}(x) - a(n-1, 0)) + xf_n(x)$$

Since $a(n, 0) = 1$ **for** $n \geq 0$ **and** $a(0, r) = 0$ **for** $r > 0$,

$$f_n(x) = \frac{f_{n-1}(x)}{1-x} = \frac{1}{(1-x)^n}.$$

So, $a(n, r)$ **is** $\binom{-n}{r}(-1)^r = \binom{n+r-1}{r}$.

• Simultaneous Linear Recurrence Relations

Ex. Solve $a_{n+1} = 2a_n + b_n$

$b_{n+1} = a_n + b_n, \quad a_0 = 1, \quad b_0 = 0$

Let $A(x) = \sum_{n=0}^{\infty} a_n x^n$ **and** $B(x) = \sum_{n=0}^{\infty} b_n x^n$

be the generating functions for $(a_0, a_1, a_2, \ldots)$ **and** $(b_0, b_1, b_2, \ldots,)$**, respectively.**

$$\sum_{n=0}^{\infty} a_{n+1}x^{n+1} = 2x\sum_{n=0}^{\infty} a_n x^n + x\sum_{n=0}^{\infty} b_n x^n$$

$$\sum_{n=0}^{\infty} b_{n+1}x^{n+1} = x\sum_{n=0}^{\infty} a_n x^n + x\sum_{n=0}^{\infty} b_n x^n$$

$$A(x) - a_0 = 2xA(x) + xB(x)$$

$$B(x) - b_0 = xA(x) + xB(x)$$

$$\Rightarrow \quad A(x) = \frac{1-x}{x^2-3x+1} = \frac{1-x}{(x-\alpha)(x-\beta)}$$

$$= \frac{-5-\sqrt{5}}{10} \cdot \frac{1}{x-\alpha} + \frac{-5+\sqrt{5}}{10} \cdot \frac{1}{x-\beta}$$

$$B(x) = \frac{x}{x^2-3x+1} = \frac{x}{(x-\alpha)(x-\beta)}$$

$$= \frac{5+3\sqrt{5}}{10} \cdot \frac{1}{x-\alpha} + \frac{5-3\sqrt{5}}{10} \cdot \frac{1}{x-\beta},$$

where $\alpha = \frac{3+\sqrt{5}}{2}$, $\beta = \frac{3-\sqrt{5}}{2}$

$$\Rightarrow \quad a_n = \frac{5+\sqrt{5}}{10} \cdot \left(\frac{3-\sqrt{5}}{2}\right)^{n+1} + \frac{5-\sqrt{5}}{10} \cdot \left(\frac{3+\sqrt{5}}{2}\right)^{n+1}$$

$$b_n = \frac{-5-3\sqrt{5}}{10} \cdot \left(\frac{3-\sqrt{5}}{2}\right)^{n+1} + \frac{-5+3\sqrt{5}}{10} \cdot \left(\frac{3+\sqrt{5}}{2}\right)^{n+1}$$

• General First-Order Linear Recurrence Relations

$$a_n = p(n)a_{n-1} + f(n)$$

The solution has the form of $a_n = a_n^h \cdot b_n$, where a_n^h is the homogeneous solution (i.e., $a_n^h = p(n)a_{n-1}^h$).

1. Find a_n^h.

2. Find b_n.

$$\begin{aligned} a_n^h \cdot b_n &= p(n)a_{n-1}^h b_{n-1} + f(n) \\ &= a_n^h b_{n-1} + f(n) \end{aligned}$$

$$\begin{aligned} \Rightarrow b_n &= b_{n-1} + \frac{f(n)}{a_n^h} \\ &= \frac{f(n)}{a_n^h} + \frac{f(n-1)}{a_{n-1}^h} + \ldots + \frac{f(1)}{a_1^h} + b_0 \\ &= \frac{f(n)}{a_n^h} + \frac{f(n-1)}{a_{n-1}^h} + \ldots + \frac{f(1)}{a_1^h} + \frac{a_0}{a_0^h} \end{aligned}$$

Ex. $a_n - \dfrac{n}{n-1} a_{n-1} = n^3, \quad a_1 = 1.$

$$a_n^h = n \cdot a_1^h$$

$(a_1^h \neq 0;$ if $a_1^h = 0,$ then $a_2^h = 0,\ a_3^h = 0,\ \ldots)$

$$(n \cdot a_1^h) \cdot b_n = \frac{n}{n-1} \cdot [(n-1) \cdot a_1^h] \cdot b_{n-1} + n^3$$

$$\begin{aligned}
\Rightarrow\ b_n &= b_{n-1} + n^2/a_1^h, \quad (b_1 = a_1/a_1^h = 1/a_1^h) \\
&= b_{n-2} + [(n-1)^2 + n^2]/a_1^h \\
&\quad \vdots \\
&= b_1 + [2^2 + \ldots + (n-1)^2 + n^2]/a_1^h \\
&= [1 + 2^2 + \ldots + (n-1)^2 + n^2]/a_1^h \\
&= \frac{n(n+1)(2n+1)}{6 \cdot a_1^h}
\end{aligned}$$

$$\Rightarrow \quad a_n = a_n^h \cdot b_n$$

$$= (n \cdot a_1^h) \cdot \frac{n(n+1)(2n+1)}{6 \cdot a_1^h}$$

$$= \frac{n^2(n+1)(2n+1)}{6}$$

(do Exercise #4)