Solutions to Exercise #9

(範圍: Graph Theory)

1. P. 518: 2. (10%)

Sol:

(a) $b \to e \to f \to e \to d$.
(b) $b \to e \to f \to g \to e \to d$.
(c) $b \to c \to d$.
(d) $b \to e \to f \to g \to e \to b$.
(e) $b \to e \to f \to g \to e \to d \to c \to b$.
(f) $b \to e \to d \to c \to b$.

2. P. 518: 4, where $\kappa(G)$ denotes the number of connected components in G. (10%)

Sol: $\kappa(G) = 2$.

Since two vertices of G are adjacent if and only if they differ in exactly two positions, a vertex of G with an even (odd) number of 1's connects to only vertices of G with even (odd) numbers of 1's.

Consider two vertices $x = (0, 0, \ldots, 0)$ and $y = (1, 0, \ldots, 0)$ of G. We have $\kappa(G) \geq 2$, because there is no path between x and y (i.e., x and y belong to two different components).

Next, we show $\kappa(G) \leq 2$. Suppose that $v \notin \{x, y\}$ is an arbitrary vertex of G. It is not difficult to see that there is a v-x (v-y) path in G if v has an even (odd) number of 1's. For example, assume $n = 5$ and $v = (1, 0, 1, 1, 1)$. A v-x path in G can be established as $v = (1, 0, 1, 1, 1) \to (1, 0, 1, 0, 0) \to (1, 0, 1, 0, 0) \to (0, 0, 0, 0, 0) = x$. Therefore, v belongs to the same component as x or y.

3. P. 519: 8. (10%)

Sol: Three guards placed at vertices a, g, i are enough.

Since there are 11 vertices in the figure and the maximal vertex degree is three, at least $\left\lceil \frac{11}{4} \right\rceil = 3$ guards are needed.

This problem is the so-called minimum vertex cover problem on a graph $G = (V, E)$, which is to determine a minimum subset V' of V such that for each edge $(u, v) \in E$, u or v belongs to V'.

4. P. 519: 9. (10%)

Sol**:**

($\Rightarrow$) Trivial.

($\Leftarrow$) Since $G - \{(a,\ b)\}$ is connected, there is an a-b path in $G - \{(a,\ b)\}$. The a-b path augmented with the edge (a, b) forms a cycle in G.

5. P. 529: 8. (10%)

Sol: (a) $P(7, 5)/2 = 1260$. (b) $P(n, m{+}1)/2$.

6. P. 529: 9. (10%)

Sol:

(a) No.

There are four vertices each of degree three in the two graphs. The four vertices in the right graph forms a cycle of length four, whereas the four in the left graph do not.

(b) Yes.

An isomorphism between the two graphs is shown below.

$a \to u$	$b \to w$	$c \to x$
$d \to y$	$e \to v$	$f \to z$

7. P. 530: 16. (10%)

Sol:

(a) $C(6, 3) \cdot 2^{\frac{3(3-1)}{2}} = 20 \cdot 2^3 = 160.$

(b) $C(6, 4) \cdot 2^{\frac{4(4-1)}{2}} = 15 \cdot 2^6 = 960.$

(c) $\displaystyle\sum_{i=1}^{6} C(6, i) \cdot 2^{\frac{i(i-1)}{2}}$, if $G = (V, E) = (\varnothing, \varnothing)$ is not considered a subgraph, or

$\displaystyle\sum_{i=1}^{6} C(6, i) \cdot 2^{\frac{i(i-1)}{2}} + 1$, if $G = (V, E) = (\varnothing, \varnothing)$ is considered a subgraph.

(d) $\displaystyle\sum_{i=1}^{n} C(n, i) \cdot 2^{\frac{i(i-1)}{2}}$, if $G = (V, E) = (\varnothing, \varnothing)$ is not considered a subgraph, or

$\displaystyle\sum_{i=1}^{n} C(n, i) \cdot 2^{\frac{i(i-1)}{2}} + 1$, if $G = (V, E) = (\varnothing, \varnothing)$ is considered a subgraph.

8. P. 537: 1. (A graph is *regular* if all its vertices have the same degree.) (10%)

Sol:

(a) $|V| \cdot 3 = 2 \cdot 9 \quad \Rightarrow \quad |V| = 6$.

(b) $|V| \cdot d(v) = 2 \cdot 15$

$\Rightarrow$ $|V| = 1, 2, 3, 5, 6, 10, 15, 30$, if loops are allowed in G, or

$|V| = 2, 3, 5, 6, 10, 15, 30$, if loops are not allowed in G.

(c) $(|V| - 2) \cdot 3 + 2 \cdot 4 = 2 \cdot 10 \quad \Rightarrow \quad |V| = 6$.

9. How to determine whether a graph G is bipartite or not? (10%)

Sol: Try to color vertices of G with two colors, say white and black, so that every pair of adjacent vertices are colored differently. The coloring succeeds if and only if G is bipartite. When G is bipartite, all white vertices constitute one partite and all black vertices constitute the other partite.

10. Show that any simple graph has an even number of vertices whose degrees are odd. (10%)

Sol: It is an immediate consequence of $\sum_{i \in V} d_i = 2 \cdot |E|$.