Solutions to Exercise #4

(範圍: Recurrence Relations)

1. P. 481: 1 (only for (a), (c) and (d)). (30%)

Sol: (a) $a_n = a_{n-1} + 2n + 1 = (a_{n-2} + 2(n-1) + 1) + 2n + 1 = \ldots = a_0 + \sum_{i=1}^{n}(2i+1)$

$= n^2 + 2n + 1 = (n+1)^2, \quad n \geq 0.$

(c) $a_n = 2a_{n-1} + 5 = 2(2a_{n-2} + 5) + 5 = \ldots = 2^n a_0 + \sum_{i=1}^{n}(5 \cdot 2^{i-1}) = 6 \cdot 2^n - 5, \quad n \geq 0.$

(d) $a_n = 2a_{n-1} + 2^n.$

$a_n^h = c(2^n)$. Let $a_n^p = kn(2^n)$.

$kn(2^n) - 2k(n-1)(2^{n-1}) = 2^{n-1}. \Rightarrow k = \frac{1}{2}.$

$a_n = a_n^h + a_n^p = c(2^n) + \frac{n}{2}(2^n).$

$a_0 = 1. \Rightarrow c = 1.$

Therefore, $a_n = 2^n + n \cdot 2^{n-1}, \quad n \geq 0.$

2. P. 481: 6. (10%)

Sol: $a_n - 6a_{n-1} + 9a_{n-2} = 3(2^{n-2}) + 7(3^{n-2}).$

$a_n^h = c_1(3^n) + c_2 n(3^n)$. Let $a_n^p = k_1(2^n) + k_2 n^2(3^n)$.

$k_1(2^n) + k_2 n^2(3^n) - 6[k_1(2^{n-1}) + k_2(n-1)^2(3^{n-1})] + 9[k_1(2^{n-2}) + k_2(n-2)^2(3^{n-2})]$
$= 3(2^{n-2}) + 7(3^{n-2})$

$\Rightarrow k_1 = 3, \quad k_2 = \frac{7}{18}.$

$a_n = a_n^h + a_n^p = (c_1 + c_2 n + \frac{7}{18}n^2)(3^n) + 3(2^n).$

$a_0 = 1$: $c_1 + 3 = 1.$

$a_1 = 4$: $3(c_1 + c_2 + \frac{7}{18}) + 6 = 4.$

$\Rightarrow\ c_1 = -2,\ \ c_2 = \dfrac{17}{18}$.

Therefore, $a_n = (-2 + \frac{17}{18}n + \frac{7}{18}n^2)(3^n) + 3(2^n),\ \ n \geq 0$.

3. P. 481: 7. (10%)

Sol: $a_n - 3a_{n-1} + 3a_{n-2} - a_{n-3} = 3 + 5(n-3)$.

$a_n^h = c_1 + c_2 n + c_3 n^2$. Let $a_n^p = k_1 n^3 + k_2 n^4$.

$k_1 n^3 + k_2 n^4 - 3[k_1(n-1)^3 + k_2(n-1)^4] + 3[k_1(n-2)^3 + k_2(n-2)^4] - [k_1(n-3)^3 + k_2(n-3)^4] = 3 + 5(n-3)$

$\Rightarrow\ k_1 = -\dfrac{3}{4},\ \ k_2 = \dfrac{5}{24}$.

Therefore, $a_n = a_n^h + a_n^p = c_1 + c_2 n + c_3 n^2 - \frac{3}{4}n^3 + \frac{5}{24}n^4,\ \ n \geq 0$.

4. P. 481: 8. (15%)

Sol: Let a_n denote the number of n-digit quaternary sequences in which no 3 appears to the right of a 0.

When the ending digit is 3, there are 3^{n-1} sequences that contain no 0.
When the ending digit is 0 (or 1 or 2), there are $3a_{n-1}$ sequences as required.

$\Rightarrow\ a_n = 3a_{n-1} + 3^{n-1},\ \ a_0 = 1,\ \ a_1 = 4$.

$a_n^h = c(3^n)$. Let $a_n^p = kn(3^n)$.

$kn(3^n) = 3kn(3^{n-1}) + 3^{n-1} \ \Rightarrow\ k = \dfrac{1}{3}$.

$a_n = a_n^h + a_n^p = c(3^n) + n(3^{n-1})$.

$a_1 = 4$: $3c + 1 = 4 \ \Rightarrow\ c = 1$.

Therefore, $a_n = 3^n + n(3^{n-1}),\ \ n \geq 0$.

5. P. 487: 1 (only for (a) and (c)). (20%)

Sol: (a) $a_n - a_{n-1} = 3^{n-1}$. Let $A(x) = \sum_{n=0}^{\infty} a_n x^n$.

$(a_n - a_{n-1})x^n = 3^{n-1}x^n$

$$\sum_{n=1}^{\infty} a_n x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = \sum_{n=1}^{\infty} 3^{n-1} x_n .$$

$$(A(x) - a_0) - xA(x) = \frac{x}{1-3x} .$$

$$A(x) = \frac{1}{1-x} + \frac{x}{(1-3x)(1-x)} = \frac{1}{2(1-x)} - \frac{1}{2(1-3x)} .$$

Therefore, $a_n = \frac{1}{2} + \frac{1}{2}(3^n), \quad n \geq 0.$

(c) $a_n - 3a_{n-1} + 2a_{n-2} = 0$. Let $A(x) = \sum_{n=0}^{\infty} a_n x^n$.

$$\sum_{n=2}^{\infty} a_n x^n - 3\sum_{n=2}^{\infty} a_{n-1} x^n + 2\sum_{n=2}^{\infty} a_{n-2} x^n = 0.$$

$$(A(x) - a_1 x - a_0) - 3x(A(x) - a_0) + 2x^2 A(x) = 0.$$

$$A(x) = \frac{1+3x}{(1-2x)(1-x)} = \frac{5}{1-2x} - \frac{4}{1-x} .$$

Therefore, $a_n = 5(2^n) - 4, \quad n \geq 0.$

6. Solve $a_n = 2na_{n-1} + n!$, $a_0 = 2$. (15%)

Sol: $a_n^h = (n!)(2^n)\, a_0^h$.

$$a_n = a_n^h \times b_n = (n!)(2^n)\, a_0^h \times b_n.$$

$$(n!)(2^n)\, a_0^h \times b_n = (2n)\, ((n-1)!)(2^{n-1})\, a_0^h \times b_{n-1} + n!.$$

$$\Rightarrow b_n = b_{n-1} + \frac{1}{(2^n) a_0^h}$$

$$= b_0 + \left(\frac{1}{a_0^h}\right) \sum_{k=1}^{n} \frac{1}{2^k}$$

$$= \frac{2}{a_0^h} + \left(\frac{1}{a_0^h}\right)(1 - \frac{1}{2^n}) \quad (b_0 = \frac{2}{a_0^h})$$

$$= \left(\frac{1}{a_0^h}\right)(3 - \frac{1}{2^n}).$$

$$
\begin{aligned}
\Rightarrow a_n &= a_n^h \times b_n \\
&= (n!)(2^n)\, a_0^h \times \left(\frac{1}{a_0^h}\right)\left(3 - \frac{1}{2^n}\right) \\
&= (n!)(3 \times 2^n - 1).
\end{aligned}
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