

# Dynamic Programming

**A sketch of a dynamic programming algorithm:**

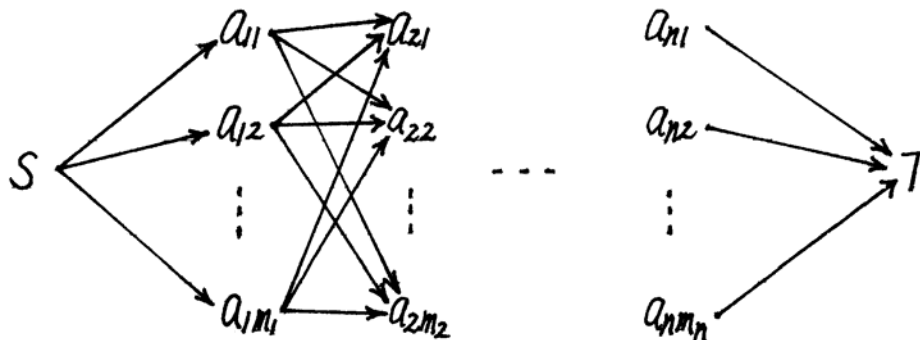
- 
- 
- 

**statements with the principle of optimality  
(expressed by recurrence relations)**

- 
- 
-

# • Principle of Optimality

Consider a multistage graph



If  $S \rightarrow a_{1r_1} \rightarrow a_{2r_2} \rightarrow \dots \rightarrow a_{nr_n} \rightarrow T$  is the shortest (longest) path between  $S$  and  $T$ , then for any  $1 \leq i < j \leq n$ , the subpath

$$a_{ir_i} \rightarrow a_{(i+1)r_{i+1}} \rightarrow \dots \rightarrow a_{jr_j}$$

is the shortest (longest) path between  $a_{ir_i}$  and  $a_{jr_j}$ .

**In general, the principle of optimality can be stated as follows.**

**“Any subpolicy of an optimum policy must itself be an optimum policy with regard to the initial and terminal states of the subpolicy.”**

**After determining an optimum policy for some initial state  $S_i$  and some terminal state  $S_t$ , we may ignore all non-optimum policies for  $(S_i, S_t)$  whenever we are looking for an optimum policy that includes  $(S_i, S_t)$  as a subpolicy.**

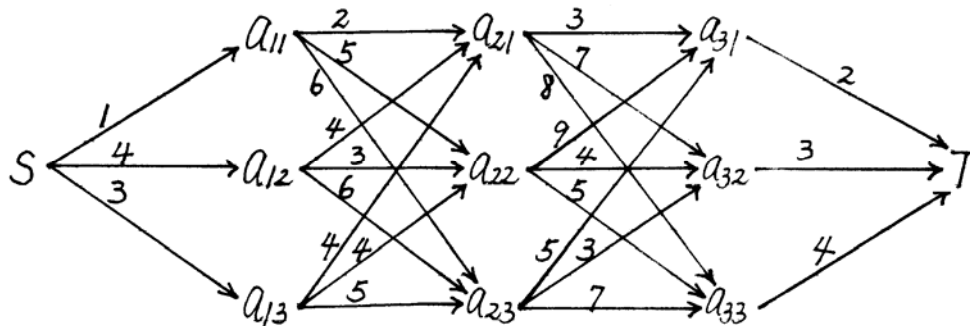
**For example, let  $l_i$  be the length of the shortest path from  $a_{1i}$  to  $T$  and  $w_i$  be the length of  $S \rightarrow a_{1i}$ .**

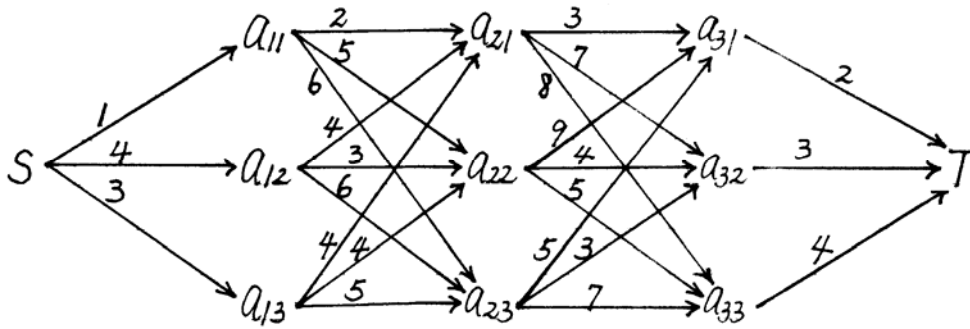
**If the shortest path from  $S$  to  $T$  is via  $a_{1i}$ , then it must include the shortest paths from  $S$  to  $a_{1i}$  and from  $a_{1i}$  to  $T$ .**

**Thus, the minimum length from  $S$  to  $T$  is  $w_i + l_i$ .**

**Based on the principle of optimality, we may devise some approaches to reducing the search space, while solving a problem by exhaustive search. Such approaches are named *dynamic programming*.**

For example, consider the following multistage graph. Three illustrative dynamic programming approaches are applied to compute the minimum length from  $S$  to  $T$ .





1. Denote by  $l_{ij}$  the minimum length from  $a_{ij}$  to  $T$ .

Step 1. Compute  $l_{31}, l_{32}, l_{33}$ .

$$(l_{31}, l_{32}, l_{33}) = (2, 3, 4) \quad (a_{3i} \rightarrow T)$$

Step 2. Compute  $l_{21}, l_{22}, l_{23}$ .

$$l_{21} = \min\{3 + l_{31}, 7 + l_{32}, 8 + l_{33}\} = 5$$

$$(a_{21} \rightarrow a_{31} \rightarrow T)$$

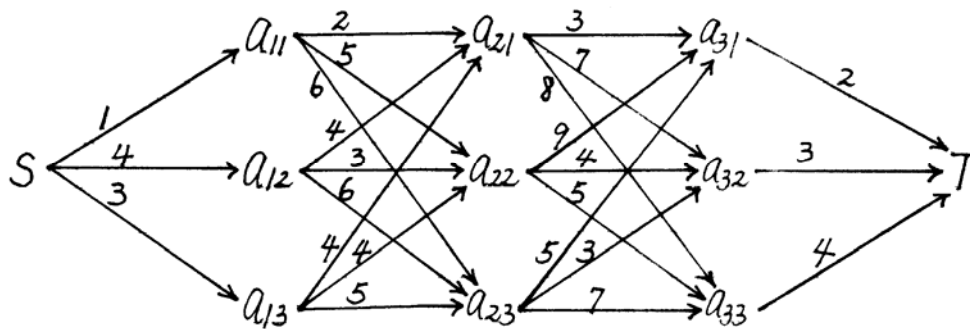
$$l_{22} = \min\{9 + l_{31}, 4 + l_{32}, 5 + l_{33}\} = 7$$

$$(a_{22} \rightarrow a_{32} \rightarrow T)$$

$$l_{23} = \min\{5 + l_{31}, 3 + l_{32}, 7 + l_{33}\} = 6$$

$$(a_{23} \rightarrow a_{32} \rightarrow T)$$





**Step 3. Compute  $l_{11}, l_{12}, l_{13}$ .**

$$l_{11} = \min\{2 + l_{21}, 5 + l_{22}, 6 + l_{23}\} = 7$$

$$(a_{11} \rightarrow a_{21} \rightarrow a_{31} \rightarrow T)$$

$$l_{12} = \min\{4 + l_{21}, 3 + l_{22}, 6 + l_{23}\} = 9$$

$$(a_{12} \rightarrow a_{21} \rightarrow a_{31} \rightarrow T)$$

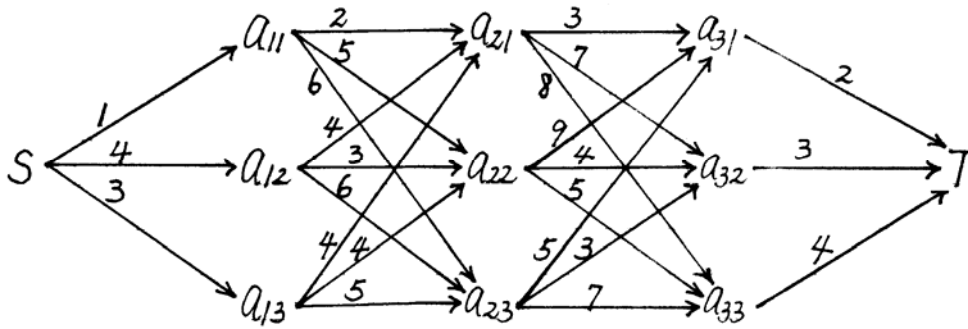
$$l_{13} = \min\{4 + l_{21}, 4 + l_{22}, 5 + l_{33}\} = 9$$

$$(a_{13} \rightarrow a_{21} \rightarrow a_{31} \rightarrow T)$$

**Step 4. The minimum length from  $S$  to  $T$  is equal**

$$\text{to } \min\{1 + l_{11}, 4 + l_{12}, 3 + l_{13}\} = 8$$

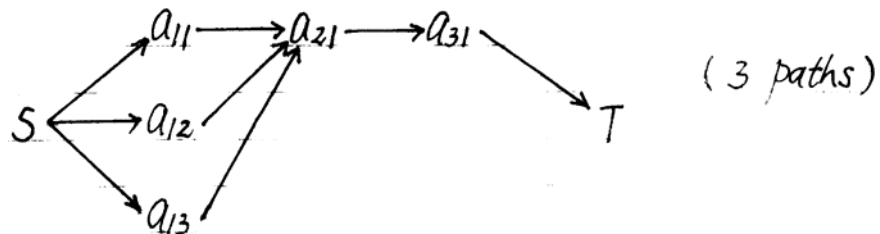
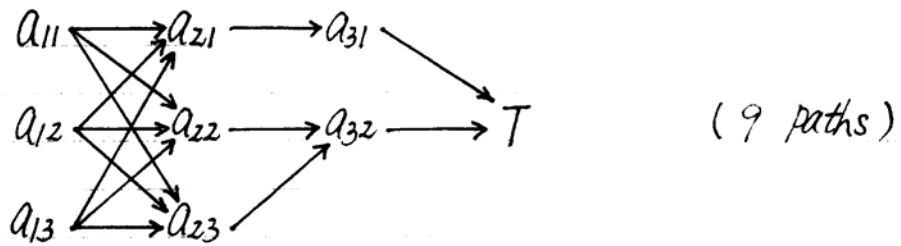
$$(S \rightarrow a_{11} \rightarrow a_{21} \rightarrow a_{31} \rightarrow T)$$

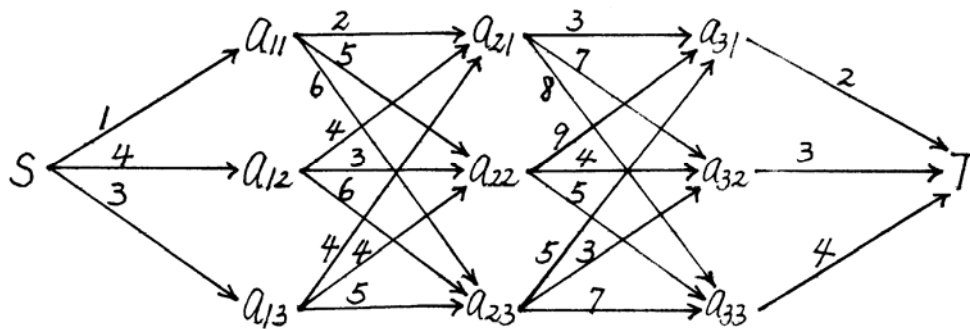


There are 24 paths computed above.

$a_{3i} \rightarrow T$   $i = 1, 2, 3$  (3 paths)

$a_{2j} \rightarrow a_{3i} \rightarrow T$   $i = 1, 2, 3$  and  $j = 1, 2, 3$  (9 paths)





2. Denote by  $p_{ij}$  the minimum length from  $S$  to  $a_{ij}$ .

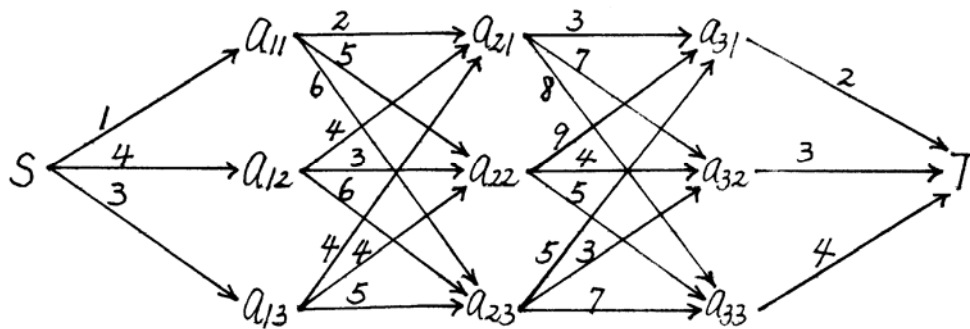
Step 1. Compute  $p_{1i}$   $i = 1, 2, 3$

Step 2. Compute  $p_{2i}$   $i = 1, 2, 3$

Step 3. Compute  $p_{3i}$   $i = 1, 2, 3$

Step 4. Compute the minimum length from  $S$  to  $T$ .

There are 24 paths computed.



**3. Compute  $l_{ij}$  and  $p_{ij}$  in opposite directions.**

**Step 1. Compute  $p_{1i}$        $i = 1, 2, 3$**

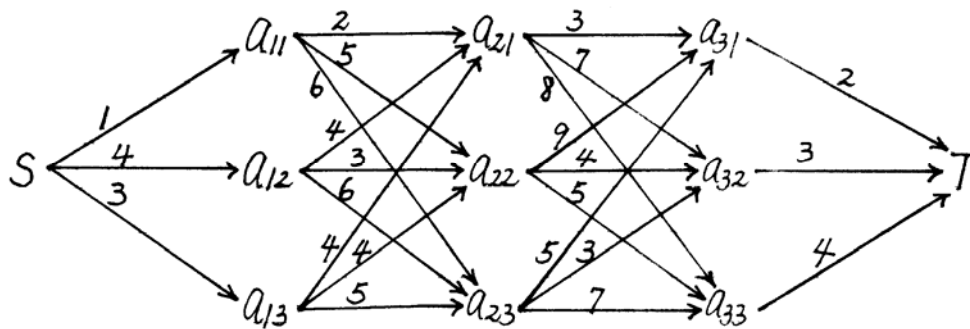
**Step 2. Compute  $p_{2i}$        $i = 1, 2, 3$**

**Step 3. Compute  $l_{3i}$        $i = 1, 2, 3$**

**Step 4. Compute  $l_{2i}$        $i = 1, 2, 3$**

**Step 5. Compute the minimum length from  $S$  to  $T$ .**

**There are 27 paths computed.**



**If we compute the minimum length from  $S$  to  $T$  by exhaustive search, then there are 27 paths (each of length 4) computed.**

**In general, the numbers of paths computed are**

**(1)  $m_1 + m_1m_2 + m_2m_3 + \dots + m_{n-1}m_n + m_n$  (approach 1)**

**(2)  $m_1 + m_1m_2 + m_2m_3 + \dots + m_{n-1}m_n + m_n$  (approach 2)**

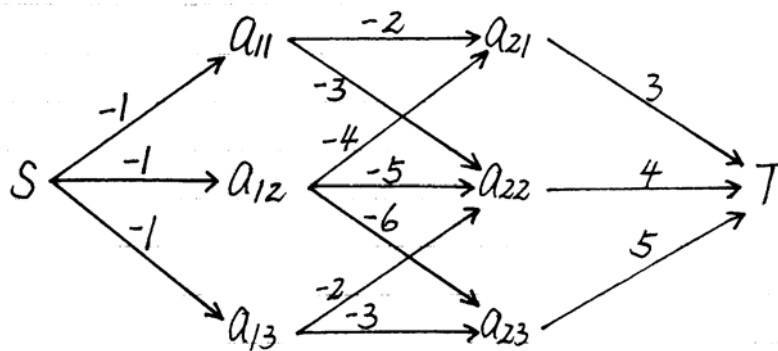
**(3)  $m_1 + m_1m_2 + m_2m_3 + \dots + m_{n-1}m_n + m_n + m_{\lceil n/2 \rceil}$   
(approach 3)**

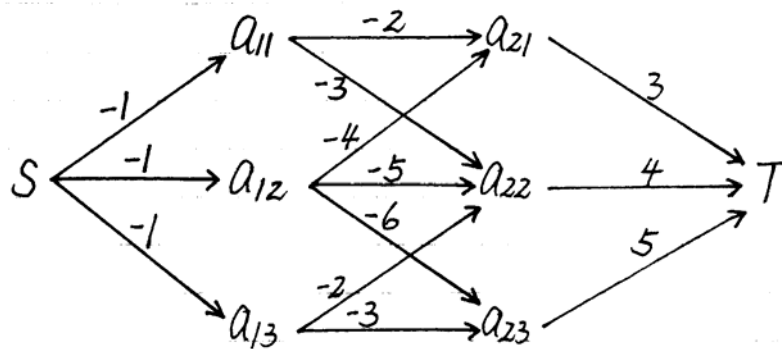
**(4)  $m_1m_2\dots m_n$  (exhaustive search)**

**$\Rightarrow$  Dynamic programming can reduce the search space considerably.**

**If the principle of optimality does not hold, then dynamic programming may fail.**

**For example, if the lengths of the arcs are allowed to be negative and the length of a path is computed as the product of the arc lengths, instead of the sum of the arc lengths, then the minimum length cannot be computed by dynamic programming.**





**The shortest path between  $S$  and  $T$  is**

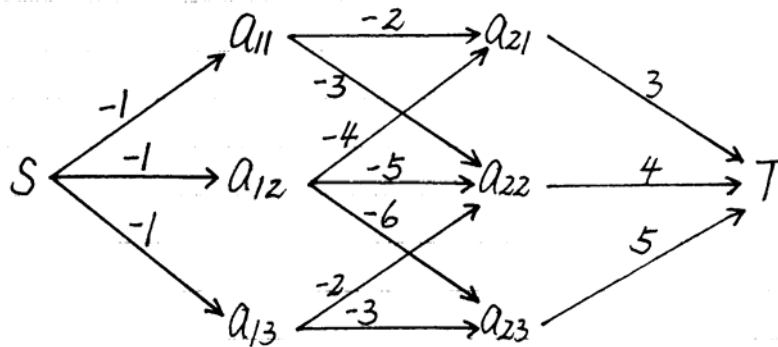
$$S \rightarrow a_{11} \rightarrow a_{21} \rightarrow T,$$

**but the shortest path between  $a_{11}$  and  $T$  is**

$$a_{11} \rightarrow a_{22} \rightarrow T.$$

**$\Rightarrow$  The principle of optimality does not hold.**





**Computing the minimum length between  $S$  and  $T$  by dynamic programming :**

$$(l_{21}, l_{22}, l_{23}) = (3, 4, 5);$$

$$l_{11} = \min\{-2 * l_{21}, -3 * l_{22}\} = -12;$$

$$l_{12} = \min\{-4 * l_{21}, -5 * l_{22}, -6 * l_{23}\} = -30;$$

$$l_{13} = \min\{-2 * l_{22}, -3 * l_{23}\} = -15;$$

$$\begin{aligned} \text{minimum length} &= \min\{-1 * l_{11}, -1 * l_{12}, -1 * l_{13}\} \\ &= 12. \end{aligned}$$

**The shortest path obtained is  $S \rightarrow a_{11} \rightarrow a_{22} \rightarrow T$ , which is incorrect.**

**Ex. The optimal parenthesization problem.**

**Given a string of  $n$  items, find an optimal parenthesization of the string.**

**A parenthesization of  $n$  items has  $n-1$  pairs of parentheses.**

**Without losing generality, let  $n$  items be  $1, 2, \dots, n$ .**

**The cost of a parenthesis pair is the sum of the enclosed numbers, and the cost of a parenthesization is the sum of the individual costs.**

$$(((1\ 2)\ 3)\ 4) \rightarrow 19$$

$$((1\ 2)\ (3\ 4)) \rightarrow 20$$

$$(1\ (2\ (3\ 4))) \rightarrow 26$$

$$((1\ (2\ 3))\ 4) \rightarrow 21$$

$$(1\ ((2\ 3)\ 4)) \rightarrow 24$$

**The optimal parenthesization is the one with the minimum cost.**

**If this problem is solved by exhaustive search, then the time required is exponential to  $n$ , for there is an exponential number of distinct parenthesizations.**

**A dynamic programming approach for this problem is proposed below.**

**The principle of optimality holds:**

**an optimal parenthesization for the whole string assures an optimal parenthesization for each of its substrings.**

**⇒ calculate optimal parenthesizations of successively larger substrings of the original string, starting from the singleton items.**

Let  $c(i, j)$  denote the minimum cost of parenthesizing the substring from the  $i$ th item to the  $j$ th item,  $1 \leq i \leq j \leq n$ .

When  $i = j$ , we have  $c(i, j) = 0$ .

Then,  $c(1, n)$ , which is the optimal value, can be computed by the following recurrence equation,

$$c(i, j) = \min_{i \leq k < j} \{c(i, k) + c(k+1, j) + \sum_{l=i}^j v_l\},$$

where  $v_l$  denotes the value of the  $l$ th item.

$$j-i = 0 : \quad c(1, 1) = c(2, 2) = c(3, 3) = c(4, 4) = 0.$$

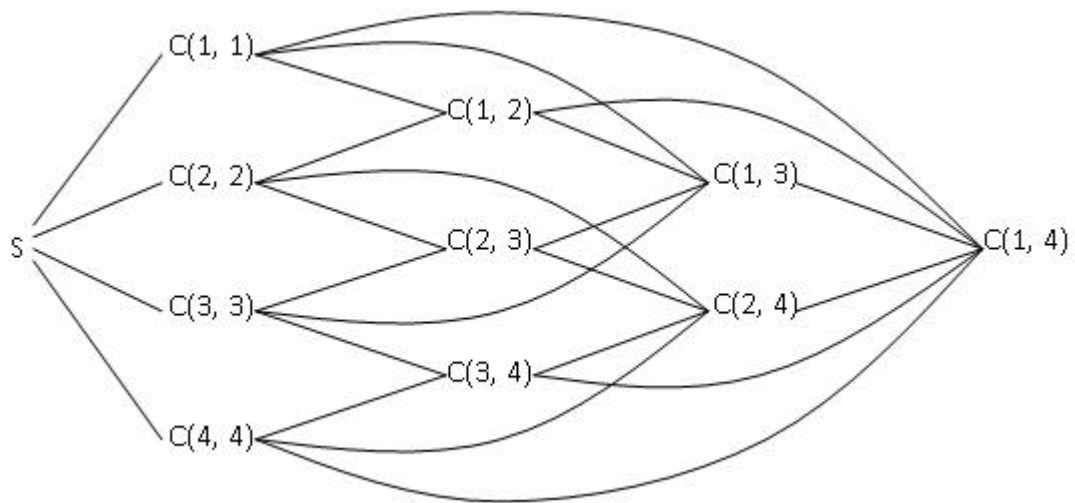
$$j-i = 1 : \quad c(1, 2) = 3, \quad c(2, 3) = 5, \quad c(3, 4) = 7.$$

$$j-i = 2 : \quad \begin{array}{rcc} \underline{(i, j)} & \underline{c(i, k) + c(k+1, j) + \sum v_l} & \underline{c(i, j)} \quad \underline{k^*} \\ (1, 3) & \mathbf{0 + 5 + 6 = 11} & \\ & \mathbf{3 + 0 + 6 = 9} & \mathbf{9} \quad \mathbf{2} \\ (2, 4) & \mathbf{0 + 7 + 9 = 16} & \\ & \mathbf{5 + 0 + 9 = 14} & \mathbf{14} \quad \mathbf{3} \end{array}$$

$$j-i = 3 : \quad \begin{array}{rcc} \underline{(i, j)} & \underline{c(i, k) + c(k+1, j) + \sum v_l} & \underline{c(i, j)} \quad \underline{k^*} \\ (1, 4) & \mathbf{0 + 14 + 10 = 24} & \\ & \mathbf{3 + 7 + 10 = 20} & \\ & \mathbf{9 + 0 + 10 = 19} & \mathbf{19} \quad \mathbf{3} \end{array}$$

The optimal value is 19, and the optimal parenthesization, which is (((1 2) 3) 4), can be obtained by the aid of  $k^*$ .

### Corresponding multistage graph :



### Time complexity :

Let  $m = j - i$ .

It takes  $O(m(n-m))$  time to compute all  $c(i, j)$ 's with  $j - i = m$ .

Hence, the total time required is

$$O\left(\sum_{m=1}^{n-1} m(n-m)\right) = O(n^3).$$

**Ex. The longest common subsequence (LCS) problem**

**Given two strings  $A = a_1a_2\dots a_n$  and  $B = b_1b_2\dots b_m$ ,  
find their longest common subsequence.**

**$A = abadc, B = adcb$**

**$\Rightarrow$  common subsequences of  $A$  and  $B$  :**

**$a, b, c, d, ab, ac, ad, dc, adc$**

**$\Rightarrow$  the longest common subsequence of  $A$  and  $B$  :**

**$adc$**

Let  $L(i, j)$  denote the length of an LCS of  $a_1a_2\dots a_i$  and  $b_1b_2\dots b_j$ , where  $1 \leq i \leq n$  and  $1 \leq j \leq m$ .

$L(n, m)$  is the length of an LCS of  $A$  and  $B$ .

The principle of optimality holds.

(1)  $a_i$  and  $b_j$  are included in an LCS ( $\Rightarrow a_i = b_j$ ):

$$L(i, j) = L(i-1, j-1) + 1.$$

(2)  $a_i$  is not included in any LCS, but  $b_j$  is included in an LCS ( $\Rightarrow a_i \neq b_j$ ):

$$L(i, j) = L(i-1, j).$$

(3)  $b_j$  is not included in any LCS, but  $a_i$  is included in an LCS ( $\Rightarrow a_i \neq b_j$ ):

$$L(i, j) = L(i, j-1).$$

(4) Neither  $a_i$  nor  $b_j$  is included in any LCS ( $\Rightarrow a_i \neq b_j$ ):

$$L(i, j) = L(i-1, j-1).$$

$$\Rightarrow L(i, j) = \max\{L(i-1, j), L(i, j-1), \\ l_{ij} + L(i-1, j-1)\},$$

where  $l_{ij} = 1$  if  $a_i = b_j$ , and  $l_{ij} = 0$  if  $a_i \neq b_j$ .

Initially,  $L(0, j) = 0$ ,  $L(i, 0) = 0$ , and  $L(0, 0) = 0$ .



For example, assume  $A = abadc$  and  $B = adcb$ .

$(i, j)$	$L(i-1, j)$	$L(i, j-1)$	$l_{ij} + L(i-1, j-1)$	$L(i, j)$
(1, 1)	0	0	<u>1</u>	<u>1</u>
(1, 2)	0	1	0	1
(2, 1)	<u>1</u>	0	0	<u>1</u>
(1, 3)	0	1	0	1
(2, 2)	1	1	1	1
(3, 1)	<u>1</u>	0	<u>1</u>	<u>1</u>
(1, 4)	0	1	0	1
(2, 3)	1	1	1	1
(3, 2)	1	1	1	1
(4, 1)	1	0	0	1
(2, 4)	1	1	2	2
(3, 3)	1	1	1	1
(4, 2)	1	1	<u>2</u>	<u>2</u>
(5, 1)	1	0	0	1
(3, 4)	2	1	1	2
(4, 3)	1	2	1	2
(5, 2)	2	1	1	2
(4, 4)	2	2	1	2
(5, 3)	2	2	<u>3</u>	<u>3</u>
(5, 4)	2	<u>3</u>	2	<u>3</u>

$$L(5, 4) \rightarrow L(5, 3) \rightarrow L(4, 2) \rightarrow L(3, 1) \rightarrow L(2, 0)$$

$$(a_5 = b_3) \quad (a_4 = b_2) \quad (a_3 = b_1)$$

or

$$L(5, 4) \rightarrow L(5, 3) \rightarrow L(4, 2) \rightarrow L(3, 1) \rightarrow L(2, 1) \rightarrow$$

$$(a_5 = b_3) \quad (a_4 = b_2)$$

$$L(1, 1) \rightarrow L(0, 0)$$

$$(a_1 = b_1)$$

**LCS :**  $(a_1, a_4, a_5) = (b_1, b_2, b_3)$  or

$$(a_3, a_4, a_5) = (b_1, b_2, b_3)$$

**The time complexity is  $O(nm)$ .**

**Ex. The knapsack problem**

$$\max \sum_{i=1}^n p_i x_i \quad (1)$$

$$\text{subject to } \sum_{i=1}^n w_i x_i \leq M$$

$$x_i \geq 0 \text{ integer } i = 1, 2, \dots, n.$$

Let  $f(k, g)$  be the maximal value of the objective function using only the first  $k$  ( $1 \leq k \leq n$ ) items with capacity limitation  $g$  ( $0 \leq g \leq M$ ), i.e.,

$$f(k, g) = \max \sum_{i=1}^k p_i x_i \quad (2)$$

$$\text{subject to } \sum_{i=1}^k w_i x_i \leq g$$

$$x_i \geq 0 \text{ integer for } 1 \leq i \leq k.$$

$f(n, M)$  is the optimal value of (1).

**The principle of optimality holds:**

**If  $(x_1^*, x_2^*, \dots, x_k^*)$  is an optimal solution to  $f(k, g)$ , then  $(x_1^*, x_2^*, \dots, x_{k-1}^*)$  is an optimal solution to  $f(k-1, g - w_k x_k^*)$ .**

$$\begin{aligned}
 f(k, g) &= \max \sum_{i=1}^k p_i x_i \\
 &\text{subject to } \sum_{i=1}^k w_i x_i \leq g \\
 &\quad x_i \geq 0 \text{ integer for } 1 \leq i \leq k \\
 &= p_k x_k^* + \sum_{i=1}^{k-1} p_i x_i^* \\
 &= p_k x_k^* + \max \sum_{i=1}^{k-1} p_i x_i \\
 &\quad \text{subject to } \sum_{i=1}^{k-1} w_i x_i \leq g - w_k x_k^* \\
 &\quad x_i \geq 0 \text{ integer for } 1 \leq i \leq k-1 \\
 &= p_k x_k^* + f(k-1, g - w_k x_k^*).
 \end{aligned}$$

## Dynamic programming 1 :

$$f(k, g) = \max\{p_k x_k + f(k-1, g - w_k x_k) \mid x_k = 0, 1, \dots, \lfloor g/w_k \rfloor\}.$$

## Time complexity :

$$\begin{aligned} T(n, M) &= O\left(\sum_{k=1}^n \sum_{g=1}^M (1 + \lfloor g/w_k \rfloor)\right) \\ &= O\left(nM + \frac{M(M+1)}{2} \sum_{k=1}^n \frac{1}{w_k}\right). \end{aligned}$$

## Dynamic Programming 2 :

Let  $(x_1^*, x_2^*, \dots, x_k^*)$  be an optimal solution to  $f(k, g)$ .

**Case 1.**  $x_k^* = 0$ .

$(x_1^*, x_2^*, \dots, x_{k-1}^*)$  is an optimal solution to  $f(k-1, g)$ .

**Case 2.**  $x_k^* \geq 1$ .

$(x_1^*, x_2^*, \dots, x_{k-1}^*, x_k^* - 1)$  is an optimal solution to  $f(k, g - w_k)$ .

$$\Rightarrow f(k, g) = \max\{f(k-1, g), p_k + f(k, g - w_k)\}.$$

**Time complexity :**  $T(n, M) = O(nM)$ .

**Ex. The 0/1 knapsack problem**

$$\max \sum_{i=1}^n p_i x_i$$

$$\text{subject to } \sum_{i=1}^n w_i x_i \leq M$$

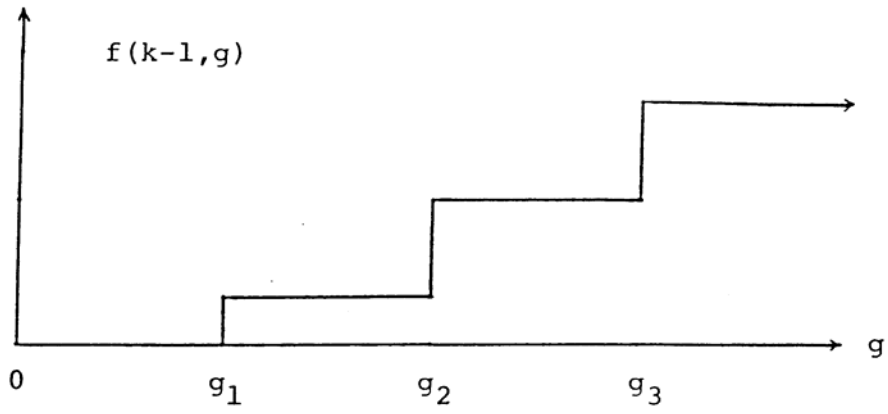
$$x_i \in \{0, 1\} \quad i = 1, 2, \dots, n.$$

$$f(k, g) = \max\{f(k-1, g), p_k + f(k-1, g - w_k)\} \quad (3)$$

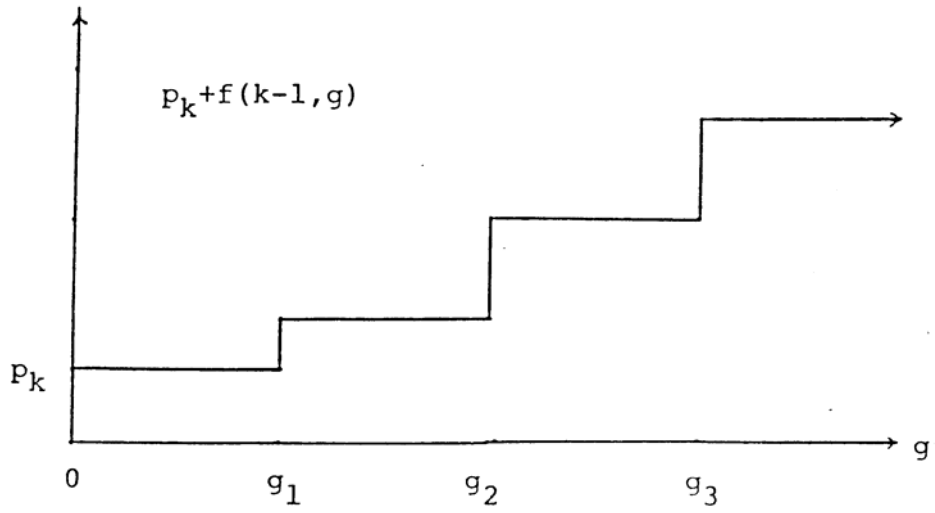
Since  $f(k, g)$  is a monotone nondecreasing step function for any fixed value of  $k$ , the function  $f$  can be evaluated for intervals  $[a, b)$ , instead of integers  $g$ , where  $f$  has the same value in  $[a, b)$ .

A geometric representation of computing  $f(k, g)$  according to (3) is below.

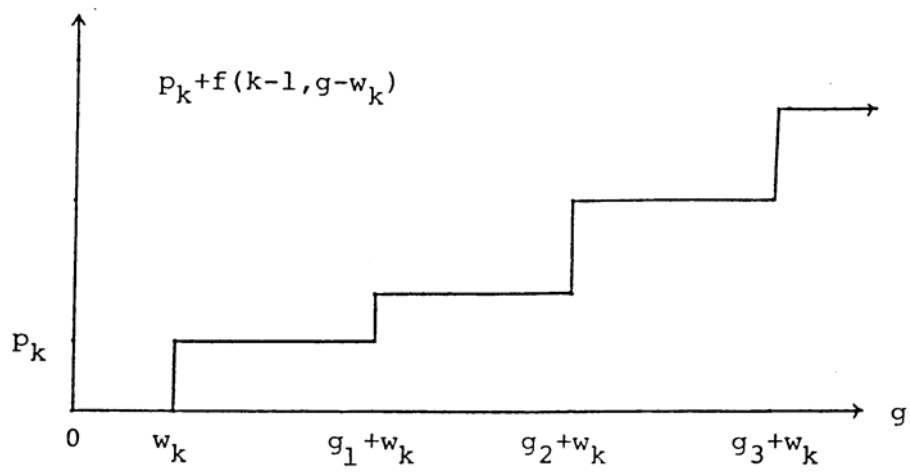




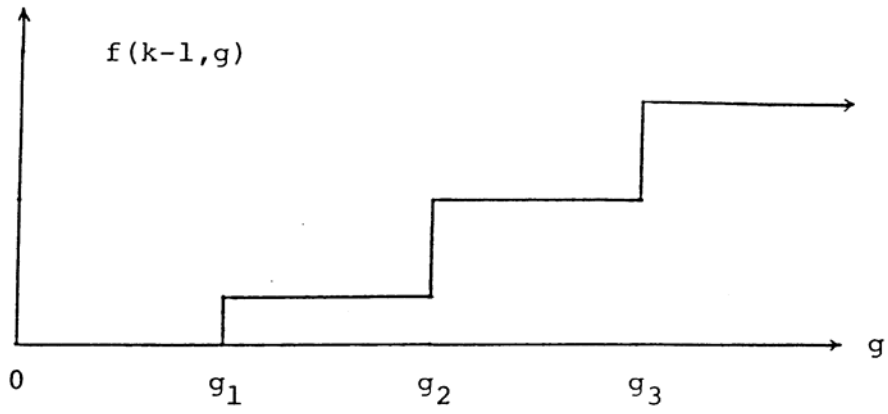
(a)



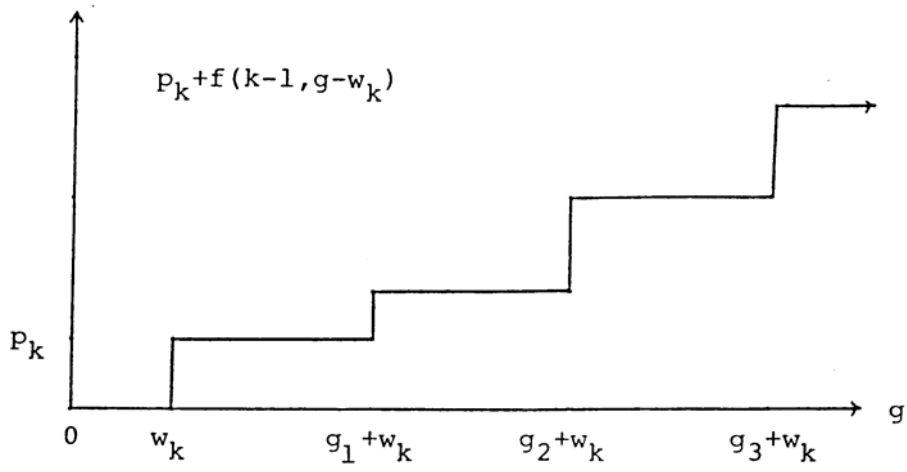
(b)



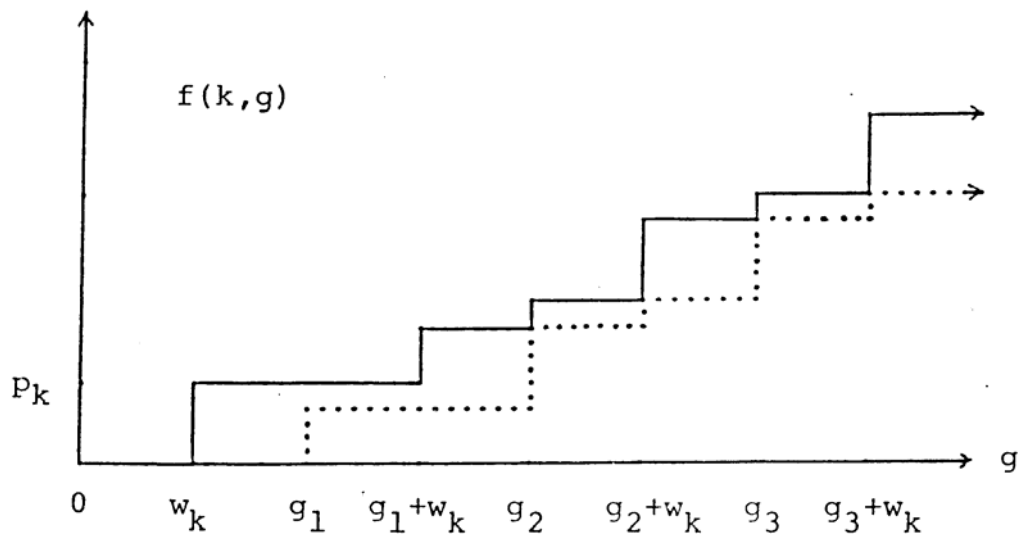
(c)



(a)



(c)



(d)

Let  $g_i$ 's be those values of  $g$  at which  $f(k-1, g)$  changes value. The possible values of  $g$  at which  $f(k, g)$  changes value are  $w_k$ ,  $g_i$ , and  $g_i + w_k$ .

If  $x$  is one of those values and  $f(k, g)$  does not change value at  $x$ , then  $x$  is called a *dominated value*. For example,  $g_1$  is a dominated value.

Denote by  $S_k$  the set of values of  $g$  at which  $f(k, g)$  changes value. Initially,  $S_1 = \{0, w_1\}$ .

For  $1 < k \leq n$ ,  $S_k$  can be determined as follows.

1.  $S_k \leftarrow S_{k-1} \cup \{w_k + d \mid d \in S_{k-1} \text{ and } w_k + d \leq M\}$ .

2. Remove dominated values from  $S_k$ .

(i)  $x \in S_{k-1}$  is a dominated value if

$$f(k-1, x) \leq p_k + f(k-1, x - w_k).$$

(ii)  $x = w_k + d$  is a dominated value if

$$f(k-1, x) \geq p_k + f(k-1, x - w_k).$$

(If  $x \in S_{k-1}$  and  $x = w_k + d$ , then  $x$  is not a dominated value.)

**An illustrative example :**

$$\begin{aligned} \text{Max} \quad & 10x_1 + 8x_2 + 5x_3 \\ \text{s.t.} \quad & 12x_1 + 9x_2 + 4x_3 \leq 16 \\ & x_1, x_2, x_3 \in \{0, 1\} \end{aligned}$$

$$S_1 = \{0, 12\}$$

<u><math>[a, b)</math></u>	<u><math>f(0, [a, b))</math></u>	<u><math>p_1 + f(0, [a - w_1, b - w_1))</math></u>	<u><math>f(1, [a, b))</math></u>
$[0, 12)$	<b>0</b>	$10 + (-\infty)$	<b>0</b>
		$(f(k, g) = -\infty, \text{ if } g < 0)$	
$[12, 17)$	<b>0</b>	<u><math>10 + 0</math></u>	<u><b>10</b></u>

$$S_2 = S_1 \cup \{9\} = \{0, 9, 12\} \text{ (no dominated value)}$$

<u><math>[a, b)</math></u>	<u><math>f(1, [a, b))</math></u>	<u><math>p_2 + f(1, [a - w_2, b - w_2))</math></u>	<u><math>f(2, [a, b))</math></u>
$[0, 9)$	<b>0</b>	$8 + (-\infty)$	<b>0</b>
$[9, 12)$	<b>0</b>	$8 + 0$	<b>8</b>
$[12, 17)$	<u><b>10</b></u>	$8 + 0$	<u><b>10</b></u>

**(9 (= 0 + 9) is not dominated, because  $f(1, 9) = 0 < 8 = p_2 + f(1, 9 - w_2) = p_2 + f(1, 0)$ ;**

**12  $\in S_1$  is not dominated, because  $f(1, 12) = 10 > 8 = p_2 + f(1, 12 - w_2) = p_2 + f(1, 3)$ .)**

$$S_2 = \{0, 9, 12\}$$

$[a, b)$	$f(1, [a, b))$	$p_2 + f(1, [a - w_2, b - w_2))$	$f(2, [a, b))$
$[0, 9)$	0	$8 + (-\infty)$	0
$[9, 12)$	0	$8 + 0$	8
$[12, 17)$	<u>10</u>	$8 + 0$	<u>10</u>

$$S_3 = S_2 \cup \{4, 13, 16\} = \{0, 4, 9, 12, 13, 16\}$$

(no dominated value)

$[a, b)$	$f(2, [a, b))$	$p_3 + f(2, [a - w_3, b - w_3))$	$f(3, [a, b))$
$[0, 4)$	0	$5 + (-\infty)$	0
$[4, 9)$	0	$5 + 0$	5
$[9, 12)$	8	$5 + 0$	8
$[12, 13)$	10	$5 + 0$	10
$[13, 16)$	10	$5 + 8$	13
$[16, 17)$	10	<u><math>5 + 10</math></u>	<u>15</u>

Optimal value :  $f(3, 16) = 15$

Optimal solution :  $(x_1^*, x_2^*, x_3^*) = (1, 0, 1)$

**Ex. The traveling salesman problem**

**A salesman wishes to visit a number of cities and then return to his starting city. Given the distances between cities, in what order should he travel so as to visit every city precisely once (except the starting city), with the minimum mileage traveled ?**

**In terms of graph theory, the aim is to find a minimum-weight hamiltonian cycle in a weighted graph.**

**Without loss of generality, denote the cities by 1, 2, ...,  $n$  and assume that the starting city is 1.**

**Let  $L(i, S)$  be the length of a shortest tour starting at city  $i$ , going through all cities in  $S$  exactly once, and ending at city 1.**

**The principle of optimality holds.**

$$L(i, S) = \min_{j \in S} \{d_{ij} + L(j, S - \{j\})\},$$

**where  $d_{ij}$  is the distance from city  $i$  to city  $j$ .**

**$L(1, \{2, 3, \dots, n\})$  is the optimal value.**



**An illustrative example :**

**Suppose that there are four cities and their distances are given by the following matrix.**

			<i>to</i>			
			1	2	3	4
<i>from</i>	1	[	0	10	15	20
	2		5	0	9	10
	3		6	13	0	12
	4		8	8	9	0
		]				

$$|S| = 0 \quad L(i, \phi) = d_{i1}$$

$ S  = 1$	$\underline{i}$	$\underline{S}$	$\underline{j}$	$\underline{d_{ij}}$	$\underline{L(j, S - \{j\})}$	$\underline{L(i, S)}$
	2	{3}	3	9	6	15
	2	{4}	4	10	8	18
	3	{2}	2	13	5	18
	3	{4}	4	12	8	20
	4	{2}	2	8	5	13
	4	{3}	3	9	6	<u>15</u>

$ S  = 1$	$\underline{i}$	$\underline{S}$	$\underline{j}$	$\underline{d_{ij}}$	$\underline{L(j, S - \{i\})}$	$\underline{L(i, S)}$
	2	{3}	3	9	6	15
	2	{4}	4	10	8	18
	3	{2}	2	13	5	18
	3	{4}	4	12	8	20
	4	{2}	2	8	5	13
	4	{3}	<u>3</u>	9	<u>6</u>	<u>15</u>

$ S  = 2$	$\underline{i}$	$\underline{S}$	$\underline{j}$	$\underline{d_{ij}}$	$\underline{L(j, S - \{i\})}$	$\underline{L(i, S)}$
	2	{3, 4}	3	9	20	
			<u>4</u>	10	<u>15</u>	<u>25</u>
	3	{2, 4}	2	13	18	
			4	12	13	25
	4	{2, 3}	2	8	15	23
			3	9	18	

$ S  = 2$	$\underline{i}$	$\underline{S}$	$\underline{j}$	$\underline{d_{ij}}$	$\underline{L(j, S - \{j\})}$	$\underline{L(i, S)}$
	2	{3, 4}	3 4	9 10	20 15	25
	3	{2, 4}	2 4	13 12	18 13	25
	4	{2, 3}	2 3	8 9	15 18	23

$ S  = 3$	$\underline{i}$	$\underline{S}$	$\underline{j}$	$\underline{d_{ij}}$	$\underline{L(j, S - \{j\})}$	$\underline{L(i, S)}$
	1	{2, 3, 4}	2 3 4	10 15 20	25 25 23	35

**Optimal value : 35**

**Optimal solution : 1 → 2 → 4 → 3 → 1**

**Ex. Maximum weight independent set in a tree**

Suppose that  $T$  is a tree whose each vertex  $i$  is associated with a positive weight  $w(i)$ . The problem is to find an independent set of  $T$  whose weight is maximum.

A vertex subset  $I$  of  $T$  is an *independent set* if no two vertices of  $I$  are adjacent.

The weight of  $I$  is  $\sum_{i \in I} w(i)$ .

The problem is NP-hard if it is defined on a general graph.

Let  $T_i$  denote the subtree of  $T$  rooted at the vertex  $i$ .

Define

$$M(i) = \max\{w(I) \mid I \text{ is an independent set of } T_i \\ \text{and } i \in I\}$$

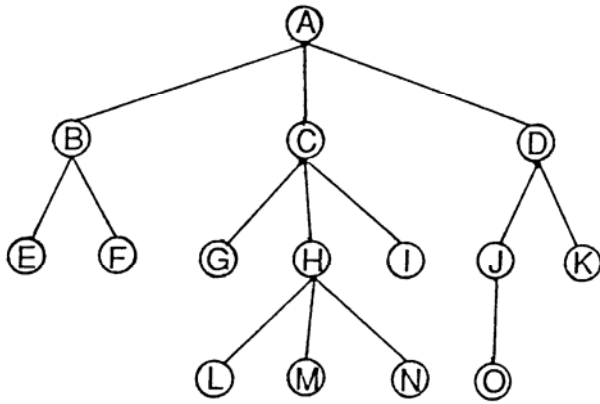
$$M'(i) = \max\{w(I) \mid I \text{ is an independent set of } T_i \\ \text{and } i \notin I\}$$

$M(i)$  ( $M'(i)$ ) is the maximum weight of the independent sets in  $T_i$  that contain  $i$  (do not contain)  $i$ .

$$M(i) = w(i) + \sum_{j \text{ is a child of } i} M'(j)$$

$$M'(i) = \sum_{j \text{ is a child of } i} \max\{M(j), M'(j)\}$$

$M(i) = w(i)$  and  $M'(i) = 0$  for each leaf vertex  $i$ .



$i$	$w(i)$	$M(i)$	$M'(i)$
A	6	53	50
B	4	4	11
C	8	23	20
D	8	10	16
E	5	5	0
F	6	6	0
G	2	2	0
H	8	8	15
I	3	3	0
J	9	9	2
K	7	7	0
L	5	5	0
M	4	4	0
N	6	6	0
O	2	2	0

The maximum weight independent set is  $\{A, E, F, G, L, M, N, I, J, K\}$ .

## **References for advanced dynamic programming algorithms.**

**(You can find more skillful dynamic programming algorithms in the following papers.)**

- (1) Knuth, “Optimal binary search trees,” *Acta Informatica*, vol. 1, 1971, 14-25.**
- (2) Yao, “Efficient dynamic programming using quadrangle inequalities,” *Proc. the ACM Symp. on the Theory of Computing*, 429-435. 1980.**
- (3) Choi and Narahari, “Algorithms for mapping and partitioning chain structured parallel computations,” *Proc. the Int’l Conf. on Parallel Processing*, vol. 1, 625-628, 1991.**
- (4) Hsu and Du, “New algorithms for the LCS problem,” *Journal of Computer and System Sciences*, vol. 29, 1984, 133-152.**
- (5) Wang, Chen, and Park, “On the set LCS and set-set LCS problems,” *Journal of Algorithms*, vol. 14, no.3, 1993, 466-477.**

## **Program Assignment 1 :**

**Write an executable program to report all longest common subsequences of two strings (and all LCS locations in the two strings).**